MATRICES

Introduction to Matrices

Definition

A rectangular arrangement of numbers, in ***m*** rows and ***n*** columns and enclosed within a bracket is called a ***matrix***.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

If ***A*** is a matrix of order **m x n** then the $\mathbf{i^{th}}$ row and the $\mathbf{j^{th}}$ column element of the matrix denoted by $\boldsymbol{a_{ij}}$. A matrix with ***m*** rows and ***n*** columns is called an **m x n** matrix. This determines the dimensions of the matrix.

Special Types of Matrices:

1. ***Square Matrix***: A matrix in which the number of rows are equal to the number of columns is called a square matrix.

$$A = \begin{bmatrix} 3 & 1 & 7 \\ 2 & -2 & 0 \\ 4 & -6 & 5 \end{bmatrix}_{3 \times 3} \qquad B = \begin{bmatrix} -1 & 5 \\ 6 & 2 \end{bmatrix}_{2 \times 2}$$

2. ***Rectangular Matrix***: A matrix in which the number of rows are not equal to the number of columns is called a rectangular matrix.

$$A = \begin{bmatrix} 1 & -2 \\ 7 & 0 \\ -9 & 3 \end{bmatrix}_{3 \times 2} \qquad B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3}$$

3. ***Diagonal Matrix***: A square matrix $\boldsymbol{A = [a_{ij}]_{n \times n}}$ is called a diagonal matrix if each of its non-diagonal element is zero. That is $\boldsymbol{a_{ij} = 0}\ if\ \boldsymbol{i \neq j}$.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

4. ***Identity Matrix***: A diagonal matrix whose diagonal elements are equal to **1** is called identity matrix .

That is $\boldsymbol{a_{ij}} = \begin{cases} \mathbf{0} & \boldsymbol{if\ i \neq j} \\ \mathbf{1} & \boldsymbol{if\ i = j} \end{cases}$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. ***Upper Triangular Matrix***: A square matrix is said to be an upper triangular if $\boldsymbol{a_{ij} = 0 \; if \; i > j}$. That is, all the elements below the diagonal are zero.

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

6. ***Lower Triangular Matrix***: A square matrix said is to be a lower triangular if $\boldsymbol{a_{ij} = 0 \; if \; i < j}$. That is, all the elements above the diagonal are zero.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -2 & 0 \\ 4 & -6 & 5 \end{bmatrix}$$

7. ***Symmetric Matrix***: A square matrix is said to be a symmetric if $\boldsymbol{a_{ij} = a_{ji}}$ for all **i** and **j**.

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & -2 & -6 \\ 4 & -6 & 5 \end{bmatrix}$$

8. ***Skew - Symmetric Matrix***: A square matrix is said to be a skew-symmetric if $\boldsymbol{a_{ij} = -a_{ji}}$ for all **i** and **j**.

$$A = \begin{bmatrix} 3 & 2 & 4 \\ -2 & -2 & -6 \\ -4 & 6 & 5 \end{bmatrix}$$

9. ***Zero Matrix***: A matrix whose all elements are zero is called a zero matrix.

$\boldsymbol{a_{ij} = 0}$ for all **i** and **j** $\quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

10. ***Row Matrix (Vector):*** A matrix consists of a single row is called as a row vector or row matrix.

$$A = \begin{bmatrix} 1 & 3 & -4 & 10 \end{bmatrix}$$

11. ***Column Matrix (Vector)***: A matrix consists of a single column is called a column vector or column matrix.

$$A = \begin{bmatrix} 2 \\ -5 \\ 9 \end{bmatrix}$$

Matrix Algebra

1. Equality of Two Matrices: Two matrices ***A*** and ***B*** are said to be equal if

(i) They are in same order.

(ii) Their corresponding elements are equal.

That is if $\boldsymbol{A} = [\boldsymbol{a_{ij}}]_{\boldsymbol{m \times n}}$ and $\boldsymbol{B} = [\boldsymbol{b_{ij}}]_{\boldsymbol{m \times n}}$ then $\boldsymbol{a_{ij} = b_{ij}}$ for example:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2x3} = \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}_{2x3} \quad if \quad \begin{matrix} a = u & b = v & c = w \\ d = x & e = y & f = z \end{matrix}$$

2. Scalar multiple of a matrix:

Let $\boldsymbol{k}$ be a scalar then scalar product of matrix $\boldsymbol{A} = [\boldsymbol{a_{ij}}]_{\boldsymbol{m \times n}}$ is given by:

$$\boldsymbol{kA} = [\boldsymbol{ka_{ij}}]_{\boldsymbol{m \times n}}$$

For example: if $k = 3$ and $A = \begin{bmatrix} 3 & 1 & 7 \\ 2 & -2 & 0 \\ 4 & -6 & 5 \end{bmatrix}_{3x3}$ then: $kA = \begin{bmatrix} 9 & 3 & 21 \\ 6 & -6 & 0 \\ 12 & -18 & 15 \end{bmatrix}_{3x3}$

3. Addition (sum) and Subtraction (difference) of Two Matrices:

The sum/difference of two matrices of the same size is a matrix with elements that are the sum/difference of the corresponding elements of the two given matrices.

Let $\boldsymbol{A} = [\boldsymbol{a_{ij}}]_{\boldsymbol{m \times n}}$ and $\boldsymbol{B} = [\boldsymbol{b_{ij}}]_{\boldsymbol{m \times n}}$ are two matrices with same size, then sum/difference of the two matrices are given by :

$$\boldsymbol{A} \pm \boldsymbol{B} = [\boldsymbol{a_{ij}}]_{\boldsymbol{m \times n}} \pm [\boldsymbol{b_{ij}}]_{\boldsymbol{m \times n}} = [\boldsymbol{a_{ij}} \pm \boldsymbol{b_{ij}}]_{\boldsymbol{m \times n}}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \pm \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} a \pm u & b \pm v & c \pm w \\ d \pm x & e \pm y & f \pm z \end{bmatrix}$$

The addition of two matrices can be done as follows:

$$\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$$

The subtraction of two matrices can be done as follows:

$$\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & -5 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 2 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -2 \\ 4 & 0 & -10 \end{bmatrix}$$

4. Multiplication (Product) of Two Matrices:

a. The product of a **1xn** row matrix and an **nx1** column matrix is equal to a **1x1** matrix given by:

$$[a_1 \quad a_2 \quad \dots \quad a_n]_{1 \times n} * \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1} = [a_1 b_1 + a_2 b_2 + \cdots + a_n b_n]_{1 \times 1}$$

Note that the number of elements in the row matrix and in the column matrix must be the same for the product to be defined.

$$[2 \quad -3 \quad 0] * \begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} = [(2)(-5) + (-3)(2) + (0)(-2)] = [-16]$$

b. Matrix Product:

The matrix product of $\boldsymbol{A} = [\boldsymbol{a_{ij}}]_{\boldsymbol{m\,x\,p}}$ and $\boldsymbol{B} = [\boldsymbol{b_{ij}}]_{\boldsymbol{p\,x\,n}}$, denoted $\boldsymbol{C} = \boldsymbol{A.B}$, is an **m x n** matrix whose element C_{ij} (in the **i**$^{\text{th}}$ row and **j**$^{\text{th}}$ column) is obtained from the product of the **i**$^{\text{th}}$ row of $\boldsymbol{A}$ and the **j**$^{\text{th}}$ column of $\boldsymbol{B}$. If the number of columns in $\boldsymbol{A}$ does not equal to the number of rows in $\boldsymbol{B}$, then the matrix product $\boldsymbol{A.B}$ is not defined.

$$[\boldsymbol{C_{ij}}]_{\boldsymbol{m\,x\,n}} = [\boldsymbol{a_{ij}}]_{\boldsymbol{m\,x\,p}} * [\boldsymbol{b_{ij}}]_{\boldsymbol{p\,x\,n}}$$

Size of product

For example:

$$C = \begin{bmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{bmatrix}_{2x3} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}_{3x2} = \begin{bmatrix} [2 \quad 3 \quad -1]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} & [2 \quad 3 \quad -1]\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \\ [-2 \quad 1 \quad 2]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} & [-2 \quad 1 \quad 2]\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \end{bmatrix}_{2x2} = \begin{bmatrix} 9 & 4 \\ -2 & -2 \end{bmatrix}$$

Ex: If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ show that $A^2 - 4A - 5I = 0$

Sol: $$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} \quad and \quad 5I = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Transpose of a Matrix

The transpose of matrix **A** $=[\boldsymbol{a_{ij}}]_{\boldsymbol{nxm}}$, written $\boldsymbol{A^T}$ is the matrix obtained by writing the rows of $\boldsymbol{A}$ in order as columns.

That is : $\boldsymbol{A^T} = [\boldsymbol{a_{ji}}]_{\boldsymbol{mxn}}$ and $(\boldsymbol{A^T})^{\boldsymbol{T}} = \boldsymbol{A}$

for example: a matrix $A = \begin{bmatrix} 3 & 1 & 7 \\ 2 & -2 & 0 \\ 4 & -6 & 5 \end{bmatrix}_{3x3}$ $then$: $A^T = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & -6 \\ 7 & 0 & 5 \end{bmatrix}_{3x3}$

A square matrix $\boldsymbol{A}$ is said to be symmetric if $\boldsymbol{A} = \boldsymbol{A^T}$

For example: $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & -2 & -6 \\ 4 & -6 & 5 \end{bmatrix}$ and $A^T = \begin{bmatrix} 3 & 2 & 4 \\ 2 & -2 & -6 \\ 4 & -6 & 5 \end{bmatrix}$

i. $\boldsymbol{A.A^T}$ and $\boldsymbol{A^T.A}$ are both symmetric.
ii. $\boldsymbol{A+A^T}$ is a symmetric matrix.
iii. $\boldsymbol{A-A^T}$ is a skew - symmetric matrix.

Ex: Verify (i) , (ii) and (iii) by using the matrix: $A=\begin{bmatrix}1 & 3 & 5\\ -3 & -5 & 10\\ 1 & 8 & 9\end{bmatrix}$

Sol:

i. $\begin{bmatrix}1 & 3 & 5\\ -3 & -5 & 10\\ 1 & 8 & 9\end{bmatrix}*\begin{bmatrix}1 & -3 & 1\\ 3 & -5 & 8\\ 5 & 10 & 9\end{bmatrix}=\begin{bmatrix}35 & 32 & 70\\ 32 & 134 & 47\\ 70 & 47 & 146\end{bmatrix}$

ii. $\begin{bmatrix}1 & 3 & 5\\ -3 & -5 & 10\\ 1 & 8 & 9\end{bmatrix}+\begin{bmatrix}1 & -3 & 1\\ 3 & -5 & 8\\ 5 & 10 & 9\end{bmatrix}=\begin{bmatrix}1 & 0 & 6\\ 0 & -10 & 18\\ 6 & 18 & 18\end{bmatrix}$

iii. $\begin{bmatrix}1 & 3 & 5\\ -3 & -5 & 10\\ 1 & 8 & 9\end{bmatrix}-\begin{bmatrix}1 & -3 & 1\\ 3 & -5 & 8\\ 5 & 10 & 9\end{bmatrix}=\begin{bmatrix}0 & 6 & 4\\ -6 & 0 & 2\\ -4 & -2 & 0\end{bmatrix}$

Determinant of a Square Matrix

Let $\boldsymbol{A}=[\boldsymbol{a_{ij}}]_{\boldsymbol{nxn}}$ be a square matrix of order **n** , then $|\boldsymbol{A}|$ is a number called determinant of the matrix $\boldsymbol{A}$.

1. Determinant of 2 x 2 Matrix

Let $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ is a 2x2 matrix, then the determinant of A is:

$$|\boldsymbol{A}|=\boldsymbol{a_{11}a_{22}-a_{21}a_{12}}$$

Ex: Find the determinant of $A=\begin{bmatrix}3 & -2\\ 7 & 4\end{bmatrix}$

Sol: $|A|=(3)(4)-(7)(-2)=26$

Ex: Find the determinant of $A=\begin{bmatrix}1+j & j2\\ -j3 & 1-j4\end{bmatrix}$

Sol: $|A|=\begin{vmatrix}(1+j) & j2\\ -j3 & (1-j4)\end{vmatrix}=(1+j)(1-j4)-(j2)(-j3)$

$$=1-j4+j-j^24+j^26$$
$$=1-j4+j-(-4)+(-6)$$
$$\boldsymbol{=-1-j3}$$

2. Determinant of 3 x 3 Matrix

The following steps have to be followed to calculate the determinant:

- Choose a row or column of $\boldsymbol{A}$.
- For every element in the chosen row or column, calculate its cofactor.
- Multiply each element of the chosen row or column by its own cofactor.
- The sum of these products is **det.(A)**.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3x3 matrix, then the determinant of A is:

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Ex: Find the determinant of $A = \begin{bmatrix} 1 & 4 & -3 \\ -5 & 2 & 6 \\ -1 & -4 & 2 \end{bmatrix}$

Sol: Using the first row:

$$|A| = 1\begin{vmatrix} 2 & 6 \\ -4 & 2 \end{vmatrix} - 4\begin{vmatrix} -5 & 6 \\ -1 & 2 \end{vmatrix} + (-3)\begin{vmatrix} -5 & 2 \\ -1 & -4 \end{vmatrix}$$

$$= (4+24) - 4(-10+6) - 3(20+2)$$

$$= 28 + 16 - 66 = \mathbf{-22}$$

Using the second column:

$$|A| = -4\begin{vmatrix} -5 & 6 \\ -1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -3 \\ -5 & 6 \end{vmatrix}$$

$$= -4(-10+6) + 2(2-3) + 4(6-15)$$

$$= 16 - 2 - 36 = \mathbf{-22}$$

An alternative way to evaluate the determinant of a 3×3 matrix is sometimes called the **basket-weave method**. This method is only applicable to 3×3 matrices. This method is illustrated in the following example:

Ex: Use the basket-weave method to calculate the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$

Sol:

$$\begin{array}{ccccc} & & \mathbf{9} & \mathbf{8} & \mathbf{20} \\ 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 4 & 2 & 1 \\ 3 & 2 & 5 & 3 & 2 \\ & & \mathbf{5} & \mathbf{24} & \mathbf{12} \end{array} \rightarrow |A| = (5 + 24 + 12) - (9 + 8 + 20) = 4$$

3. Determinant of 4 x 4 Matrix

The procedure of finding the determinant of a 3x3 matrix may be extended to find the determinant of a 4x4 matrix as illustrated in following example:

Ex: Find the determinant of $A = \begin{bmatrix} 2 & 3 & 0 & 5 \\ 1 & 3 & 2 & 4 \\ 5 & 3 & 4 & 1 \\ 4 & 2 & 3 & 5 \end{bmatrix}$

$$|A| = 2\begin{vmatrix} 3 & 2 & 4 \\ 3 & 4 & 1 \\ 2 & 3 & 5 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 1 \\ 4 & 3 & 5 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & 4 \\ 5 & 3 & 1 \\ 4 & 2 & 5 \end{vmatrix} - 5\begin{vmatrix} 1 & 3 & 2 \\ 5 & 3 & 4 \\ 4 & 2 & 3 \end{vmatrix}$$

$$= 2(51 - 26 + 4) - 3(17 - 42 - 4) - 5(1 + 3 - 4) = 145$$

Elementary Row Operations

The elementary row operations can be used to find the determinant of a square matrix $\boldsymbol{A}$ by reducing the matrix to an upper triangular matrix, then the determinant is the product of the diagonal elements as illustrated in the following example.

Ex: Find the determinant of the matrix: $A = \begin{bmatrix} 2 & 4 & 2 & 1 \\ 4 & 3 & 0 & -1 \\ -6 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

Sol:

$$\begin{bmatrix} 2 & 4 & 2 & 1 \\ 4 & 3 & 0 & -1 \\ -6 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{matrix} -2R1 + R2 \\ \\ 3R1 + R3 \end{matrix} \rightarrow \begin{bmatrix} 2 & 4 & 2 & 1 \\ 0 & -5 & -4 & -3 \\ 0 & 12 & 8 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix} R2 \leftrightarrow R4$$

$$\rightarrow \begin{bmatrix} 2 & 4 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 12 & 8 & 3 \\ 0 & -5 & -4 & -3 \end{bmatrix} \begin{matrix} -12R2 + R3 \\ \\ 5R2 + R4 \end{matrix} \rightarrow \begin{bmatrix} 2 & 4 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -21 \\ 0 & 0 & 1 & 7 \end{bmatrix} R3 \leftrightarrow R4$$

$$\rightarrow \begin{bmatrix} 2 & 4 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & -4 & -21 \end{bmatrix} 4R3 + R4 \rightarrow \begin{bmatrix} 2 & 4 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

The determinant of A is: $|A| = 2 * 1 * 1 * 7 = 14$

We can find the determinant by another way as follows:

$$|A| = \begin{vmatrix} 2 & 4 & 2 & 1 \\ 0 & -5 & -4 & -3 \\ 0 & 12 & 8 & 3 \\ 0 & 1 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} -5 & -4 & -3 \\ 12 & 8 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$|A| = 2[-5(16 - 3) + 4(24 - 3) - 3(12 - 8)] = 2(7) = 14$$

Properties of the Determinant:

- The determinant of a matrix A and its transpose are equal $|\boldsymbol{A}| = |\boldsymbol{A^T}|$.
- If $\boldsymbol{A}$ has a row or a column of zeros then $|\boldsymbol{A}| = \boldsymbol{0}$.
- If $\boldsymbol{A}$ has two identical rows or columns then $|\boldsymbol{A}| = \boldsymbol{0}$.
- If $\boldsymbol{A}$ is a triangular (upper or lower) matrix or a diagonal matrix then $|\boldsymbol{A}|$ is the product of the diagonal elements: $\mathbf{det}(\boldsymbol{A}) = \boldsymbol{a_{11}a_{22}a_{33}} \dots \dots \boldsymbol{a_{nn}}$
- If $\boldsymbol{A}$ is a square matrix of order **n** and **k** is a scalar then $|\boldsymbol{kA}| = \boldsymbol{k}|\boldsymbol{A}|$.
- $\boldsymbol{det}(\boldsymbol{A}.\boldsymbol{B}) = \boldsymbol{det}(\boldsymbol{A}) \,.\, \boldsymbol{det}\,(\boldsymbol{B})$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size.
- $\boldsymbol{det}(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{det}(\boldsymbol{A}) + \boldsymbol{det}\,(\boldsymbol{B})$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size.

Singular Matrix

If $\boldsymbol{A}$ is square matrix of order **n**, the $\boldsymbol{A}$ is called singular matrix when $|\boldsymbol{A}| = \boldsymbol{0}$ and non- singular otherwise.

For example $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ is a singular matrix $|A| = 2 * 3 - 6 * 1 = 0$

Minor, Cofactor and Adjoint Matrices

Let $\boldsymbol{A} = [\boldsymbol{a_{ij}}]_{\boldsymbol{nxn}}$ is a square matrix, then:

- $\boldsymbol{M_{ij}}$ denote a sub matrix of $\boldsymbol{A}$ obtained by deleting its $\mathbf{i^{th}}$ row and $\mathbf{j^{th}}$ column. The determinant $|\boldsymbol{M_{ij}}|$ is called the minor of the element $\boldsymbol{a_{ij}}$ of $\boldsymbol{A}$.
- The cofactor of $\boldsymbol{a_{ij}}$ denoted by $\boldsymbol{C_{ij}}$ and is equal to $\boldsymbol{C_{ij}} = (-\mathbf{1})^{\boldsymbol{i+j}}|\boldsymbol{M_{ij}}|$.
- The transpose of the matrix of cofactors of the element $\boldsymbol{a_{ij}}$ of $\boldsymbol{A}$ denoted by $\boldsymbol{adj\ A}$ is called adjoint of matrix $\boldsymbol{A}$.

Ex: Find the minor, cofactor and adjoint of the matrix $= \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{bmatrix}$

Sol:

The matrix of minor is: $\begin{bmatrix} \begin{vmatrix} 0 & 7 \\ -3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 7 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 \\ -3 & -2 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 21 & -11 & -6 \\ -11 & -5 & -13 \\ 28 & 23 & -8 \end{bmatrix}$

The matrix of cofactors $= \begin{bmatrix} 21 & 11 & -6 \\ 11 & -5 & 13 \\ 28 & -23 & -8 \end{bmatrix}$

$adj\ of\ A = \begin{bmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{bmatrix}$

Inverse of a Square Matrix

The inverse of a square matrix $\boldsymbol{A}$, denoted by $\boldsymbol{A^{-1}}$ is given by:

$$\boldsymbol{A^{-1}} = \frac{\boldsymbol{adjoint\ of\ A}}{\boldsymbol{determinant\ of\ A}}\ \frac{\boldsymbol{adjA}}{\boldsymbol{|A|}}$$

- $\boldsymbol{A.A^{-1} = I} \quad and \quad \boldsymbol{A^{-1}.A = I}$
- $\boldsymbol{(A^{-1})^{-1} = A}$
- $\boldsymbol{det(A^{-1}) = \frac{1}{det(A)}}$
- Any matrix is said to be invertible if its determinant is not equal to zero.
- $\boldsymbol{(A.B)^{-1} = B^{-1}.A^{-1}} \quad and \quad \boldsymbol{det[(A.B)^{-1}] = det(A^{-1}).det(B^{-1})}$
- If $\boldsymbol{A}$ is a diagonal matrix, $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ then: $A^{-1} = \begin{bmatrix} 1/a_{11} & 0 & 0 \\ 0 & 1/a_{22} & 0 \\ 0 & 0 & 1/a_{33} \end{bmatrix}$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2 x 2 matrix then the inverse of matrix $\boldsymbol{A}$ is:

$$\boldsymbol{A^{-1} = \frac{1}{a.d - b.c}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Ex: Find the inverse of $A = \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$

Sol: $det(A) = 3 * 4 - 1 * 5 = 7 \ then\ A^{-1} = \frac{1}{7}\begin{bmatrix} 4 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{-5}{7} \\ \frac{-1}{7} & \frac{3}{7} \end{bmatrix}$

H.W For the previous example, prove that: $A^{-1}.A = I$

Ex: Determine the inverse of the matrix: $A = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{bmatrix}$

Sol:

The matrix of cofactors $= \begin{bmatrix} 21 & 11 & -6 \\ 11 & -5 & 13 \\ 28 & -23 & -8 \end{bmatrix}$

The transpose of the matrix of cofactors is:

$$adj\ of\ A = \begin{bmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{bmatrix}$$

The determinant of A is $|A| = -2(-8-3) + 0 - 7(-9-4) = 22 + 91 = 113$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{113}\begin{bmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{bmatrix}$$

Solution of Simultaneous Linear System of Equations

A third order system of linear equations has the following form:

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n & = & b_1 \cdots\cdots 1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n & = & b_2 \cdots\cdots 2 \\ \vdots \quad \vdots \quad \ddots \quad \vdots & = & \vdots \\ \vdots \quad \vdots \quad \ddots \quad \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n & = & b_n \cdots\cdots n \end{array}$$

The matrix form of the system is:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad OR \quad \boldsymbol{AX = B}$$

1. Solution Using Cramer's Rule

The following steps have to be followed:

- Find $\boldsymbol{D}$,the determinant of, $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{bmatrix}$ $\qquad \boldsymbol{D \neq 0}$
- Find $\boldsymbol{D_1}$,the determinant of, $\begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n1} & \cdots & a_{nn} \end{bmatrix}$ replace the **1st** column by vector **B**.
- Find $\boldsymbol{D_2}$,the determinant of, $\begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}$ replace the **2nd** column by vector **B**.
- Find $\boldsymbol{D_n}$,the determinant of, $\begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n1} & \cdots & b_n \end{bmatrix}$ replace the **nth** column by vector **B**.
- The solution is : $\boldsymbol{x_1} = \frac{\boldsymbol{D_1}}{\boldsymbol{D}}$, $\boldsymbol{x_2} = \frac{\boldsymbol{D_2}}{\boldsymbol{D}}$ …… $and\ \boldsymbol{x_n} = \frac{\boldsymbol{D_n}}{\boldsymbol{D}}$

Ex: Solve the following system of linear equations:

$$x + y + z = 4$$
$$2x - 3y + 4z = 33$$
$$3x - 2y - 2z = 2$$

Sol:

$$AX = B \;\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 33 \\ 2 \end{bmatrix}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & -2 \end{vmatrix} = 1(6+8) - 1(-4-12) + 1(-4+9) = 35$$

$$D_x = \begin{vmatrix} 4 & 1 & 1 \\ 33 & -3 & 4 \\ 2 & -2 & -2 \end{vmatrix} = 4(6+8) - 1(-66-8) + 1(-66+6) = 70$$

$$D_y = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 33 & 4 \\ 3 & 2 & -2 \end{vmatrix} = 1(-66-8) - 4(-4-12) + 1(4-99) = -105$$

$$D_z = \begin{vmatrix} 1 & 1 & 4 \\ 2 & -3 & 33 \\ 3 & -2 & 2 \end{vmatrix} = 1(-6+66) - 1(4-99) + 4(-4+9) = 175$$

$$\therefore\, x = \frac{D_x}{D} = \frac{70}{35} = 2 \qquad y = \frac{D_y}{D} = \frac{-105}{35} = -3 \qquad z = \frac{D_z}{D} = \frac{175}{35} = 5$$

2. Solution Using Matrix Inverse

The system of linear equations can be written as:

$$\boldsymbol{AX} = \boldsymbol{B}$$

Multiply both sides by $\boldsymbol{A^{-1}}$, to give:

$$\boldsymbol{A^{-1}AX} = \boldsymbol{A^{-1}B}$$

But $\boldsymbol{A^{-1}.A = I}$ and $\boldsymbol{I.X = X}$ then, the solution is:

$$\boldsymbol{X = A^{-1}.B}$$

Ex: Solve the following system of linear equations:

$$x + 2y = 4$$

$$3x - 5y = 1$$

Sol:

$$AX = B \ \rightarrow \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{(1)(-5)-(3)(2)} \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} = \frac{-1}{11} \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}.B = \frac{-1}{11} \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{-1}{11} \begin{bmatrix} -22 \\ -11 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x = 2 \quad and \quad y = 1$$

Ex: Solve the simultaneous equations:

$$x_1 - 2x_2 + x_3 = 3$$
$$2x_1 + x_2 - x_3 = 5$$
$$3x_1 - x_2 + 2x_3 = 12$$

Sol:

$$AX = B \ \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix}$$

$$|A| = 1(2-1) + 2(4+3) + 1(-2-3) = 1 + 14 - 5 = 10$$

The cofactor of $A = \begin{bmatrix} 1 & -7 & -5 \\ 3 & -1 & -5 \\ 1 & 3 & 5 \end{bmatrix} \ \rightarrow \ adj\ A = \begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}$

$$\therefore \ A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{10} \begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{10} \begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 30 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$Then \ x_1 = 3\,, \qquad x_2 = 1 \ and \ x_3 = 2$$

Tutorial Sheet

Matrix Algebra:

Matrices A to K are given below:

$A = \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix}$

$C = \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$

$D = \begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$

$E = \begin{pmatrix} 3 & 6 & 2 \\ 5 & -3 & 7 \\ -1 & 0 & 2 \end{pmatrix}$

$F = \begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix}$ $G = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

$H = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ $J = \begin{pmatrix} 4 \\ -11 \\ 7 \end{pmatrix}$ $K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$

In Problems 1 to12, perform the matrix operation stated.

1. $A+B$ Ans. $\left[\begin{pmatrix} 8 & 1 \\ -5 & 13 \end{pmatrix}\right]$

2. $D+E$ Ans. $\left[\begin{pmatrix} 7 & -1 & 8 \\ 3 & 1 & 7 \\ 4 & 7 & -2 \end{pmatrix}\right]$

3. $A-B$ Ans. $\left[\begin{pmatrix} -2 & -3 \\ -3 & 1 \end{pmatrix}\right]$

4. $A+B-C$ Ans. $\left[\begin{pmatrix} 9.3 & -6.4 \\ -7.5 & 16.9 \end{pmatrix}\right]$

5. $5A+6B$ Ans. $\left[\begin{pmatrix} 45 & 7 \\ -26 & 71 \end{pmatrix}\right]$

6. $2D+3E-4F$ Ans. $\left[\begin{pmatrix} 4.6 & -5.6 & -7.6 \\ 17.4 & -16.2 & 28.6 \\ -14.2 & 0.4 & 17.2 \end{pmatrix}\right]$

7. $A \times H$ Ans. $\left[\begin{pmatrix} -11 \\ 43 \end{pmatrix}\right]$

8. $A \times B$ Ans. $\left[\begin{pmatrix} 16 & 0 \\ -27 & 34 \end{pmatrix}\right]$

9. $A \times C$ Ans. $\left[\begin{pmatrix} -6.4 & 26.1 \\ 22.7 & -56.9 \end{pmatrix}\right]$

10. $D \times J$ Ans. $\left[\begin{pmatrix} 135 \\ -52 \\ -85 \end{pmatrix}\right]$

11. $E \times K$ Ans. $\left[\begin{pmatrix} 5 & 6 \\ 12 & -3 \\ 1 & 0 \end{pmatrix}\right]$

12. $D \times F$ Ans. $\left[\begin{pmatrix} 55.4 & 3.4 & 10.1 \\ -12.6 & 10.4 & -20.4 \\ -16.9 & 25.0 & 37.9 \end{pmatrix}\right]$

13. Show that $A \times C \neq C \times A$

Ans. $\left[\begin{array}{l} A \times C = \begin{pmatrix} -6.4 & 26.1 \\ 22.7 & -56.9 \end{pmatrix} \\ C \times A = \begin{pmatrix} -33.5 & -53.1 \\ 23.1 & -29.8 \end{pmatrix} \\ \text{Hence they are not equal} \end{array}\right]$

Determinant

1. Use the definition of the determinant to evaluate the determinant of the given matrix.

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ 5 & 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 4 & 3 \\ 2 & 3 & 1 & 4 \end{bmatrix}$ (f) $\begin{bmatrix} 3 & 2 & 0 & 1 \\ 5 & 0 & 3 & 2 \\ 0 & 4 & 0 & 2 \\ 3 & 1 & 0 & 2 \end{bmatrix}$ (g) $\begin{bmatrix} 5 & 4 & 3 & 1 \\ 0 & 4 & 1 & 2 \\ 3 & 1 & 2 & 4 \\ 0 & 2 & 3 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 3 & 0 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 2 & 3 & 1 & 0 \\ 4 & 1 & 0 & 3 \end{bmatrix}$

2. Use the basket-weave method to evaluate the determinants 1(a), 1(b), 1(c) and 1(d).

3. Evaluate the determinants of the following matrices by inspection.

(a) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 5 & 9 \\ 0 & 6 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 0 \\ 1 & 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

4. Use elementary row operations to evaluate the determinants of the following matrices.

(a) $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 4 & 3 \\ 3 & 6 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 & 5 \\ 2 & 6 & 8 \\ 1 & 3 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 8 & 4 \\ 2 & 6 & 8 \\ 3 & 9 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 8 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 3 & 5 & 4 \\ 1 & 3 & 0 & 2 \\ 2 & 4 & 6 & 4 \\ 3 & 6 & 3 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 4 & 7 \\ 2 & 4 & 1 & 5 \\ 3 & 6 & 9 & 2 \end{bmatrix}$ (g) $\begin{bmatrix} 3 & 1 & 2 & 4 \\ 6 & 5 & 3 & 7 \\ 9 & 3 & 2 & 1 \\ 6 & 2 & 4 & 5 \end{bmatrix}$

Matrix Inverse

Find the inverse of the following matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Solution of Linear System of Equations

In Problems 1 to 5 use **matrices** to solve the simultaneous equations given.

1. $3x + 4y = 0$
   $2x + 5y + 7 = 0$ Ans.$[x = 4,\ y = -3]$

2. $2p + 5q + 14.6 = 0$
   $3.1p + 1.7q + 2.06 = 0$
   Ans.$[\,p = 1.2,\ q = -3.4]$

3. $x + 2y + 3z = 5$
   $2x - 3y - z = 3$
   $-3x + 4y + 5z = 3$
   Ans.$[x = 1,\ y = -1,\ z = 2]$

4. $3a + 4b - 3c = 2$
   $-2a + 2b + 2c = 15$
   $7a - 5b + 4c = 26$
   Ans.$[a = 2.5,\ b = 3.5,\ c = 6.5]$

5. $p + 2q + 3r + 7.8 = 0$
   $2p + 5q - r - 1.4 = 0$
   $5p - q + 7r$ Ans $3.5 = 0$
   Ans.$[\,p = 4.1,\ q = -1.9,\ r = -2.7]$

6. In two closed loops of an electrical circuit, the currents flowing are given by the simultaneous equations:
   $I_1 + 2I_2 + 4 = 0$
   $5I_1 + 3I_2 - 1 = 0$
   Use matrices to solve for I_1 and I_2
   Ans.$[I_1 = 2,\ I_2 = -3]$

7. Kirchhoff's laws are used to determine the current equations in an electrical network and show that
   $$i_1 + 8i_2 + 3i_3 = -31$$
   $$3i_1 - 2i_2 + i_3 = -5$$
   $$2i_1 - 3i_2 + 2i_3 = 6$$
   Use determinants to solve for i_1, i_2 and i_3.
   Ans.$[i_1 = -5,\ i_2 = -4,\ i_3 = 2]$

8. Applying mesh-current analysis to an a.c. circuit results in the following equations:
   $$(5 - j4)I_1 - (-j4)I_2 = 100\angle 0^\circ$$
   $$(4 + j3 - j4)I_2 - (-j4)I_1 = 0$$
   Solve the equations for I_1 and I_2
   Ans. $\begin{bmatrix} I_1 = 10.77\angle 19.23^\circ\text{A}, \\ I_2 = 10.45\angle -56.73^\circ\text{A} \end{bmatrix}$