Transmission lines

Electrical Department-3rd Stage

Lecture Two

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❑ INPUT IMPEDANCE OF TRANSMISSION LINE

❖ Consider a transmission line of length ℓ ,characterized by γ and Z_0, connected to a load Z_L as shown in Figure 1 (a).

❖ Looking into the line, the generator sees the line with the load as an input impedance Zin. It is our intention in this section to determine the input impedance, the standing wave ratio (SWR), and the power flow on the line.

❖ Let the transmission line extend from z = 0 at the generator to z = ℓ , at the load. First of all, we need the voltage and current waves in eqs. (1.1) and (1.2), that is,

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad \text{......................(1.1)}$$

$$I_s(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} \text{............................(1.2)}$$

Where $Z_0 = \frac{V_0^+}{I_0^+}$

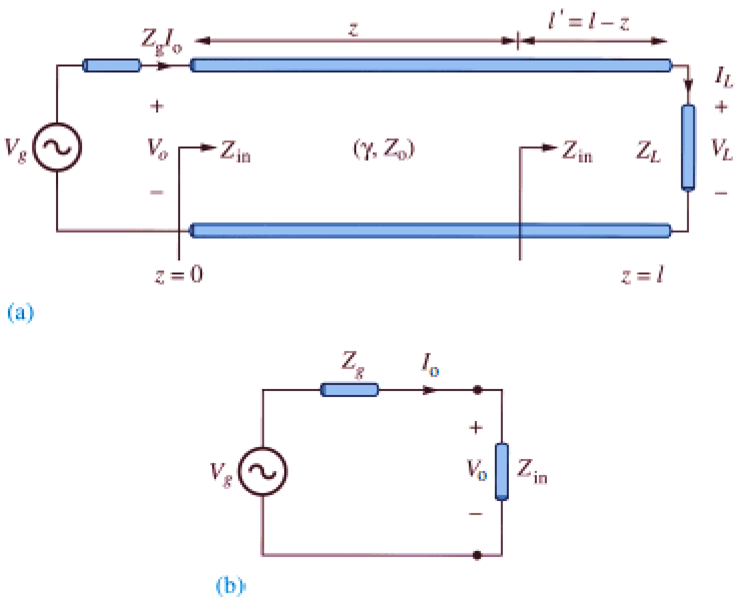

FIGURE 1 (a) Input impedance due to a line terminated by a load. (b) Equivalent circuit for finding Vo and Io in terms of Zin at the input.

❖ To find V_0^+ and V_0^- , the terminal conditions must be given. For example, if we are given the conditions at the input, say

$$V_0 = V(z=0), I_0 = I(z=0) \text{......................(1.3)}$$

substituting these into eqs. (1.1) and (1.2) results in

$$V_0^+ = \frac{1}{2}(V_0 + I_0\, Z_0) \text{........................(1.4a)}$$
$$V_0^- = \frac{1}{2}(V_0 - I_0\, Z_0) \text{........................(1.4b)}$$

If the input impedance at the input terminals is Zin, the input voltage Vo and the input Current Io are easily obtained from Figure 1(b) as

$$V_0 = \frac{Z_{in}}{Z_{in}+Z_g} V_g, \quad I_0 = \frac{V_g}{Z_{in}+Z_g} \dots\dots\dots\dots\dots..(1.5)$$

On the other hand, if we are given the conditions at the load, say

$$V_L = V(z = l), I_L = I(z = l) \text{......................(1.6)}$$

Substituting these into eqs. (1.1) and (1.2) gives

$$V_0^+ = \frac{1}{2}(V_L + I_L\, Z_0)e^{\gamma l} \text{........................(1.7a)}$$
$$V_0^- = \frac{1}{2}(V_L - I_L\, Z_0)\, e^{-\gamma l} \text{ ....(1.7b)}$$

Next, we determine the input impedance $Z_{in}=\frac{V_s(z)}{I_s(z)}$ at any point on the line. At the generator, for example, eqs. (1.1) and (1.2) yield

$$Z_{in}=\frac{V_s(z)}{I_s(z)} = \frac{Z_0\left(V_0^{+}+V_0^{-}\right)}{\left(V_0^{+}-V_0^{-}\right)} \dots\dots\dots\dots\dots\dots\dots\dots\dots. (1.8)$$

Substituting eq. (1.7) into (1.8) and utilizing the fact that

$$\frac{e^{\gamma l}+e^{-\gamma l}}{2}=\cosh\gamma l,\ \frac{e^{\gamma l}-e^{-\gamma l}}{2}=\sinh\gamma l \ \dots\dots\dots(1.9a) \qquad \text{Or} \quad \tanh\gamma l=\frac{\sinh\gamma l}{\cosh\gamma l}= \frac{e^{\gamma l}-e^{-\gamma l}}{e^{\gamma l}+e^{-\gamma l}} \dots\dots\dots(1.9b) \qquad \text{We get}$$

$$\boldsymbol{Z_{in}= Z_0\left[\frac{Z_L+Z_0\tanh\gamma l}{Z_0+Z_L\tanh\gamma l}\right]}\dots\dots\dots\dots(1.10) \text{ --for (lossy transmission line)}$$

Although eq. (1.10) has been derived for the input impedance Zin at the generation end, it is a general expression for finding Zin at any point on the line.

To find Z_{in} at a distance l' from the load as in Figure 1(a), we replace l by l' .

A formula for calculating the hyperbolic tangent of a complex number, required in eq. (1.10), is found in Appendix I.

For a lossless line, $\gamma = J\beta$, tanh$j\beta l$= $jtan\beta l$, and Zo =Ro, so eq. (1.10) becomes

$$\boldsymbol{Z_{in}= Z_0 \left[\frac{Z_L+jZ_0 tan\beta l}{Z_0+jZ_L tan\beta l}\right]} \ldots\ldots\ldots\ldots\ldots\ldots(1.11)$$ **For lossless transmission line**

showing that the input impedance varies periodically with distance l from the load. The quantity βl in eq. (1.11) is usually referred to as the electrical length of the line and can be expressed in degrees or radians.

❑ Reflection coefficient (Γ)

We now define Γ_L as the voltage reflection coefficient (at the load). The reflection coefficient Γ_L is the ratio of the voltage reflection wave to the incident wave at the load; that is,

$$\boldsymbol{\Gamma_L=\frac{V_0^- e^{\gamma l}}{V_0^+ e^{-\gamma l}}} \ldots\ldots\ldots\ldots\ldots\textbf{(1.12)}$$

Substituting V_0^+ and V_0^- in eq. (1.7) into eq. (1.12) and incorporating $V_L = Z_L I_L$ gives

$$\boldsymbol{\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}} \quad \dots \dots (\mathbf{1.13})$$

The voltage reflection coefficient at any point on the line is the ratio of the reflected voltage wave to that of the incident wave.

That is,

$$\boldsymbol{\Gamma\,(Z) = \frac{V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z}} = \frac{V_0^-}{V_0^+} e^{2\gamma z}}$$

But $z = l - l'$. Substituting and combining with eq. (1.12), we get

$$\boldsymbol{\Gamma\,(Z) = \frac{V_0^-}{V_0^+} e^{2\gamma l} e^{-2\gamma l'}} \quad \dots \dots (\mathbf{1.14})$$

The current reflection coefficient at any point on the line is the negative of the voltage reflection coefficient at that point.

Thus, the current reflection coefficient at the load is $\boldsymbol{I_0^- e^{\gamma l} / I_0^+ {-e^{\gamma l}} = -\Gamma_L}$

❑ Standing Wave Ratio (SWR)

Standing wave ratio (SWR) is the ratio of the maximum magnitude or amplitude of a standing wave to its minimum magnitude. It indicates whether there is an impedance mismatch between the load and the internal impedance on a radio frequency (RF) transmission line, or waveguide.

we define the *standing wave ratio as*

$$s = \frac{V_{max}}{V_{min}} = \frac{I_{max}}{I_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \quad \ldots\ldots..(1.15a)$$

$$|\Gamma_L| = \frac{s-1}{s+1} \quad ....(1.15b)$$

It is easy to show that $I_{max} = V_{max}/Z_o$ and $I_{min} = V_{min}/Z_o$. The input impedance Z_{in} in eq. (1.11) has maxima and minima that occur, respectively, at the maxima and minima of the voltage standing wave. It can also be shown that

$$|Z_{in}|_{max} = \frac{V_{max}}{I_{min}} = sZ_0 \quad \ldots\ldots\ldots.. \quad (1.16a)$$

and

$$|Z_{in}|_{min} = \frac{V_{min}}{I_{max}} = \frac{Z_0}{s} \quad ....(1.16b)$$

As a way of demonstrating these concepts, consider a lossless line with characteristic impedance of Zo=50 V. For the sake of simplicity, we assume that the line is terminated in a pure resistive load ZL =100 V and the voltage at the load is 100 V (rms). The conditions on the line are displayed in Figure 2. Note from Figure 2 that conditions on the line repeat themselves every half-wavelength.

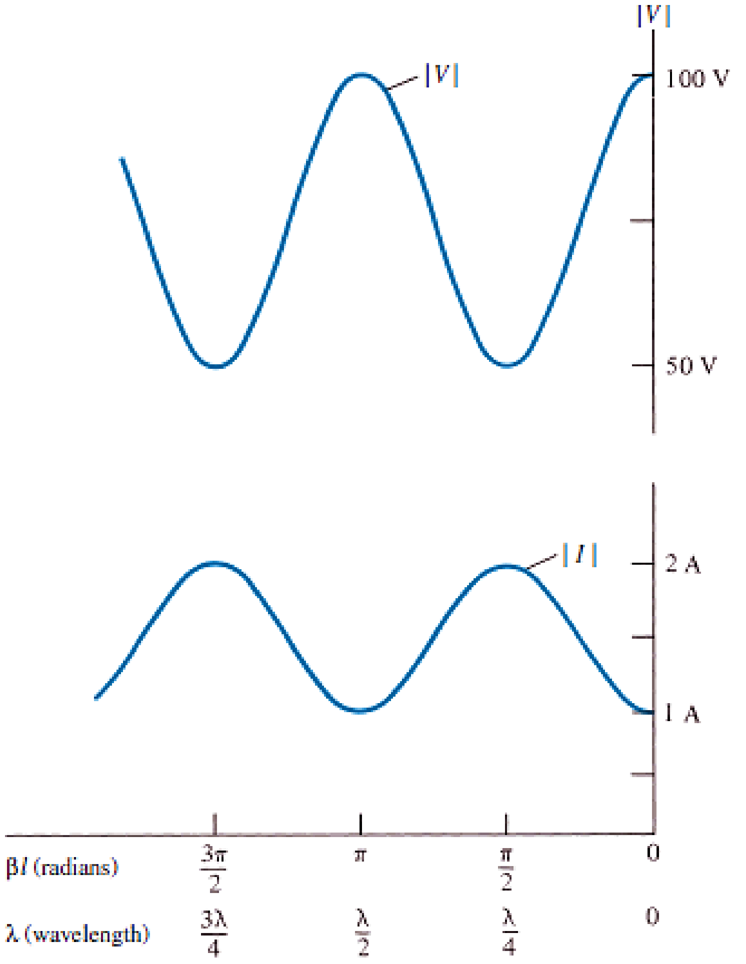

FIGURE 2 Voltage and current standing wave patterns on a lossless line terminated by a resistive load.

❑ The average input power

The average input power at a distance l from the load is given by the below equation

$$P_{av} = \frac{1}{2}\, Re\,[V_s(l) I^*{}_s(l)]$$

where the factor $\frac{1}{2}$ is needed because we are dealing with the peak values instead of the rms values.

Assuming a lossless line we substitute eqs. (1.1) and (1.2) to obtain

$$P_{\text{ave}} = \frac{1}{2}\text{Re}\left[V_o^+\left(e^{j\beta\ell} + \Gamma e^{-j\beta\ell}\right)\frac{V^{+*}}{Z_o}\left(e^{-j\beta\ell} - \Gamma^* e^{j\beta\ell}\right)\right]$$

$$= \frac{1}{2}\text{Re}\left[\frac{|V_o^+|^2}{Z_o}\left(1 - |\Gamma|^2 + \Gamma e^{-2j\beta\ell} - \Gamma^* e^{2j\beta\ell}\right)\right]$$

Since the last two terms together become purely imaginary, we have

$$\boxed{P_{\text{ave}} = \frac{|V_o^+|^2}{2Z_o}\left(1 - |\Gamma|^2\right)} \qquad \text{............(1.18)}$$

The first term is the incident power *Pi*, while the second term is the reflected power *Pr*. Thus eq. (1.18) may be written as

$$\boldsymbol{P_t = P_i - P_r}$$

where Pt is the input or transmitted power and the negative sign is due to the negative going wave (since we take the reference direction as that of the voltage/current traveling toward the right). We should notice from eq. (1.18) that the power is constant and does not depend on l, since it is a lossless line. Also, we should notice **that maximum power is delivered to the load when Γ=0 , as expected.**

A. Shorted Line (Z_L = 0)

For this case this eq. $Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 tan\beta l}{Z_0 + jZ_L tan\beta l}\right]$ will become

$$Z_{SC} = Z_{SC}|_{Z_L=0} = jZ_0 tan\beta l \quad .... (1.19)$$

Also, from eqs. (1.13) and (1.15) $\Gamma_L = -1 , s = \infty$

We notice from eq. (1.19) that Zin is a pure reactance, which could be capacitive or inductive depending on the value of l. The variation of Zin with l is shown in Figure 11.8(a).

B. Open-Circuited Line (Z_L = ∞)

In this case, this eq. $Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 tan\beta l}{Z_0 + jZ_L tan\beta l}\right]$ becomes

$$Z_{oc} = \lim_{Z_L \to \infty} Z_{in} = \frac{Z_o}{j \tan \beta \ell} = -jZ_o \cot \beta \ell \quad ....(1.20)$$

Also, from eqs. (1.13) and (1.15)

$$\Gamma_L = 1\ , s = \infty$$

The variation of Zin with l is shown in Figure 11.8(b). Notice from eqs. (1. 19) and (1.20) that

$$Z_{sc}Z_{oc} = Z_0^2 \ldots\ldots\ldots\ldots..(1.21)$$

C. Matched Line (ZL = Zo)

The most desired case from the practical point of view is the matched line i.e., ZL = Zo. For this case, this eq. $Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 tan\beta l}{Z_0 + jZ_L tan\beta l}\right]$ reduces to

Zin = Zo……..(1.22)

Also, from eqs. (1.13) and (1.15)

$$\Gamma_L = 0\ , s = 1$$

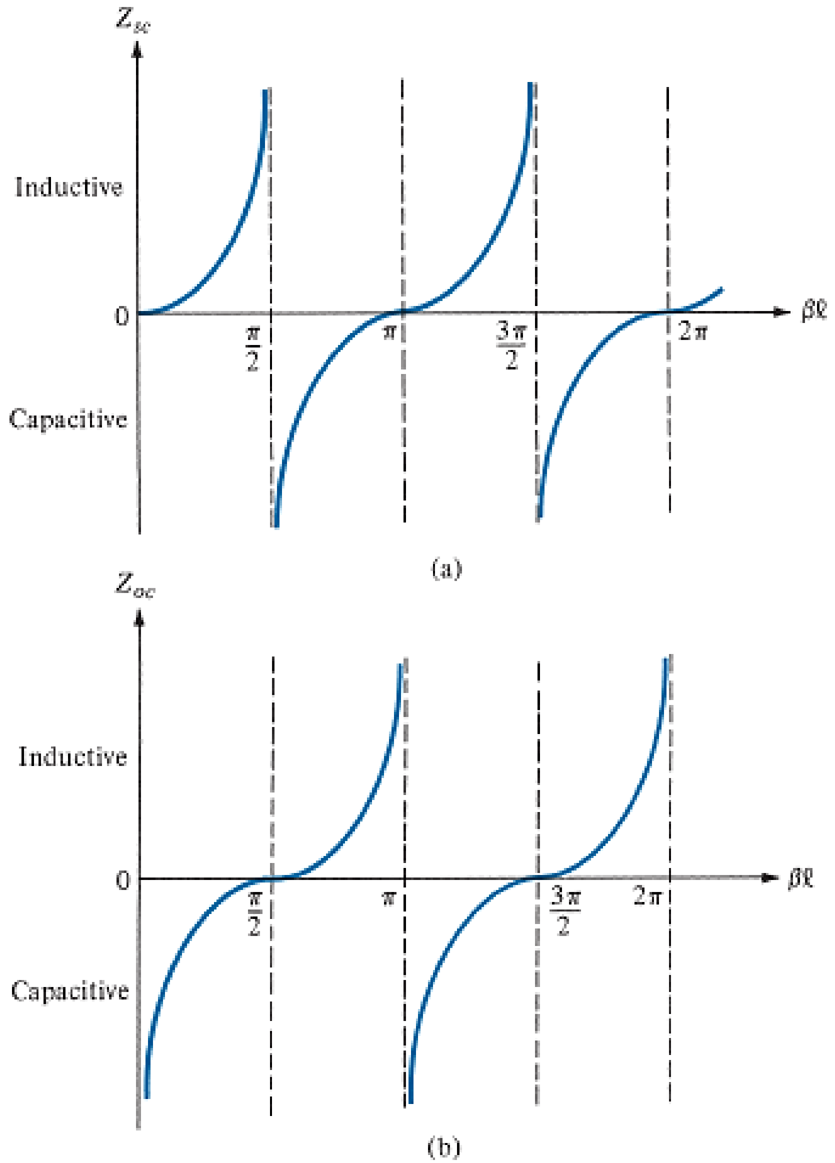

FIGURE 1.3 Input impedance of a lossless line: **(a)** when shorted, **(b)** when open.

that is, $\boldsymbol{V_0^-} = \mathbf{0}$; the whole wave is transmitted, and there is no reflection. The incident power is fully absorbed by the load. Thus maximum power transfer is possible when a transmission line is matched to the load.

EXAMPLE 1

A certain transmission line 2 m long operating at $\omega = 10^6\ rad/s$ and $\alpha = 8\frac{dB}{m}$, $\beta = 1\frac{rad}{m}$ and Zo=60+j40 Ω. If the line is connected to a source of $10\angle 0^0$ V, Zg=40 Ω and terminated by a load of 20 + j50 Ω, determine

(a) The input impedance
(b) The sending-end current
(c) The current at the middle of the line

Solution:

(a) Since 1 Np = 8.686 dB,

$$\alpha = \frac{8}{8.686} = 0.921\ \text{Np/m}$$

$$\gamma = \alpha + j\beta = 0.921 + j1\ /\text{m}$$

$$\gamma\ell = 2(0.921 + j1) = 1.84 + j2$$

Using the formula for $\tanh(x + jy)$ in Appendix A.3, we obtain

$$\tanh \gamma\ell = 1.033 - j0.03929$$

$$Z_{\text{in}} = Z_o\left(\frac{Z_L + Z_o \tanh \gamma\ell}{Z_o + Z_L \tanh \gamma\ell}\right)$$

$$= (60 + j40)\left[\frac{20 + j50 + (60 + j40)(1.033 - j0.03929)}{60 + j40 + (20 + j50)(1.033 - j0.03929)}\right]$$

$$Z_{\text{in}} = 60.25 + j38.79\ \Omega$$

(b) The sending-end current is $I(Z{=}0) = I_0$ From eq. (1.5),

$$I(z=0) = \frac{V_g}{Z_{in} + Z_g} = \frac{10}{60.25 + j38.79 + 40}$$
$$= 93.03\angle{-21.15^\circ} \text{ mA}$$

(c) To find the current at any point, we need $\boldsymbol{V_0^+}$ and $\boldsymbol{V_0^-}$. But

$$I_o = I(z=0) = 93.03\angle{-21.15^\circ} \text{ mA}$$

$$V_o = Z_{in}I_o = (71.66\angle{32.77^\circ})(0.09303\angle{-21.15^\circ}) = 6.667\angle{11.62^\circ} \text{ V}$$

From eq. (1.4),

$$V_o^+ = \frac{1}{2}(V_o + Z_oI_o)$$
$$= \frac{1}{2}[6.667\angle{11.62^\circ} + (60 + j40)(0.09303\angle{-21.15^\circ})] = 6.687\angle{12.08^\circ}$$

$$V_o^- = \frac{1}{2}(V_o - Z_oI_o) = 0.0518\angle{260^\circ}$$

At the middle of the line, $z = \ell/2$, $\gamma z = 0.921 + j1$. Hence, the current at this point is

$$I_s(z=\ell/2) = \frac{V_o^+}{Z_o}e^{-\gamma z} - \frac{V_o^-}{Z_o}e^{\gamma z}$$
$$= \frac{(6.687e^{j12.08^\circ})e^{-0.921-j1}}{60 + j40} - \frac{(0.0518e^{j260^\circ})e^{0.921+j1}}{60 + j40}$$

Note that $j1$ is in radians and is equivalent to $j57.3^\circ$. Thus,

$$I_s(z=\ell/2) = \frac{6.687e^{j12.08^\circ}e^{-0.921}e^{-j57.3^\circ}}{72.1e^{j33.69^\circ}} - \frac{0.0518e^{j260^\circ}e^{0.921}e^{j57.3^\circ}}{72.1e^{33.69^\circ}}$$
$$= 0.0369e^{-j78.91^\circ} - 0.001805e^{j283.61^\circ}$$
$$= 6.673 - j34.456 \text{ mA}$$
$$= 35.10\angle{281^\circ} \text{ mA}$$

❑ THE SMITH CHART

The Smith chart is the most commonly used of the graphical techniques. It is basically a graphical indication of the impedance of a transmission line and of the corresponding reflection coefficient as one moves along the line

Used for calculations of transmission line characteristics such as $\boldsymbol{\Gamma_L}$, s, and Zin.

By assuming that the transmission line to which the Smith chart will be applied is **lossless** $(\boldsymbol{Z_0 = R_0})$

The Smith chart is constructed within a circle of unit radius $(|\boldsymbol{\Gamma_L}| \leq \mathbf{1})$ as shown in Figure 1.4.

The construction of the chart is based on the relation in eq. (1. 13); that is

$$\boldsymbol{\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}} \quad \dots \dots (\mathbf{1.13})$$

or

$$\boldsymbol{\Gamma = |\Gamma| \angle\theta_\Gamma = \Gamma_r + j\Gamma_i} \dots\dots\dots\dots\dots\dots\dots\dots\dots (1.23)$$

where $\boldsymbol{\Gamma_r}$ and $\boldsymbol{\Gamma_i}$ are the real and imaginary parts of the reflection coefficient $\boldsymbol{\Gamma}$.

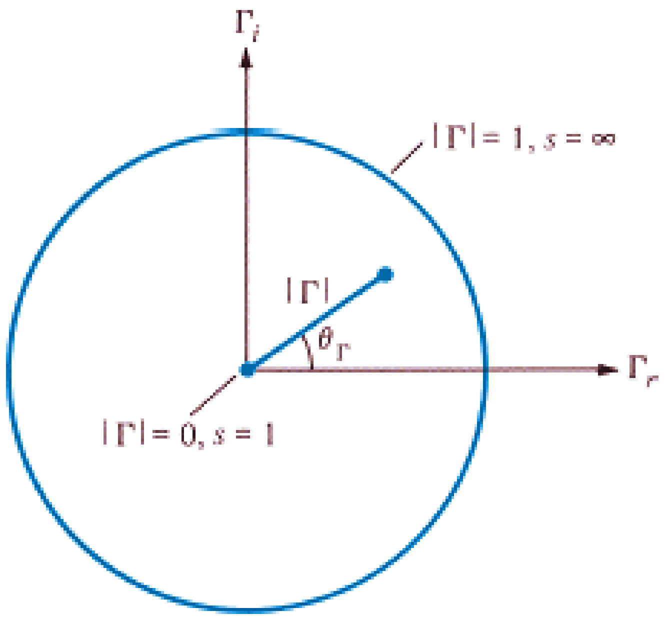

FIGURE 1.4 Unit circle on which the Smith chart is constructed.

Instead of having separate Smith charts for transmission lines with different characteristic impedances (e.g., Zo = 60, 100, 120 Ω), we prefer to have just one that can be used for any line. We achieve this by using a normalized chart in which all impedances are normalized with respect to the characteristic impedance Zo of the particular line under consideration. For the load impedance Z_L , for example, the normalized impedance Z_L is given by

$$z_L = \frac{Z_L}{Z_o} = r + jx \qquad ....(1.24)$$

Substituting eq. (1. 24) into eqs. (1. 13) and (1. 23) gives

$$\Gamma = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1} \qquad ....(1.25)$$

or

$$z_L = r + jx = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i} \qquad ....(1.26)$$

Normalizing and equating real and imaginary components, we obtain

$$r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \qquad ....(1.27)$$

$$x = \frac{2\,\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \qquad ....(1.28)$$

Rearranging terms in eqs. (1. 27 & 28) leads to

$$\boxed{\left[\Gamma_r - \frac{r}{1+r}\right]^2 + \Gamma_i^2 = \left[\frac{1}{1+r}\right]^2} \quad \text{....(1.29)} \quad \text{and} \quad \boxed{[\Gamma_r - 1]^2 + \left[\Gamma_i - \frac{1}{x}\right]^2 = \left[\frac{1}{x}\right]^2} \quad \text{....(1.30)}$$

Each of eqs. (11) and (1) is similar to

$$(x-h)^2 + (y-k)^2 = a^2$$

which is the general equation of a circle of radius a, centered at (h, k). Thus eq. (11.50) is an *r-circle (resistance circle)* with

$$\text{center at } (\Gamma_r, \Gamma_i) = \left(\frac{r}{1+r}, 0\right)$$

$$\text{radius} = \frac{1}{1+r}$$

TABLE 1 Radii and Centers of r-Circles for Typical Values of r

Normalized Resistance (r)	Radius $\left(\frac{1}{1+r}\right)$	Center $\left(\frac{r}{1+r}, 0\right)$
0	1	(0, 0)
1/2	2/3	(1/3, 0)
1	1/2	(1/2, 0)
2	1/3	(2/3, 0)
5	1/6	(5/6, 0)
∞	0	(1, 0)

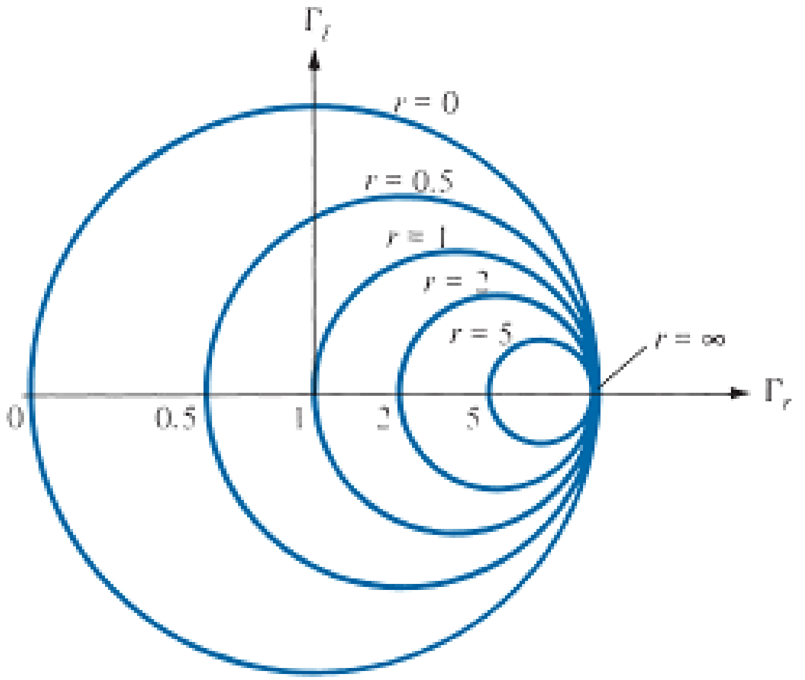

FIGURE 5 Typical r-circles for $r = 0, 0.5, 1, 2, 5, \infty$.

Appendix I

HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\sin jx = j \sinh x, \quad \cos jx = \cosh x$$

$$\sinh jx = j \sin x, \quad \cosh jx = \cos x$$

$$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\sinh (x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y$$

$$\cosh (x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y$$

$$\tanh (x \pm jy) = \frac{\sinh 2x}{\cosh 2x + \cos 2y} \pm j \frac{\sin 2y}{\cosh 2x + \cos 2y}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$\sin (x \pm jy) = \sin x \cosh y \pm j \cos x \sinh y$$

$$\cos (x \pm jy) = \cos x \cosh y \mp j \sin x \sinh y$$