Lect. -6-

Series :

المتسلسلة :- هي مجموعة من القيم بينها عمليات جمع وتتبع قاعدة معينة تسمى a_n .

$$a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n = \sum_{n=1}^{\infty} a_n = S_n$$

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_{n-1} = a_1 + a_2 + a_3 + \cdots + a_{n-1}$$

$$\boxed{S_n = S_{n-1} + a_n}$$

Ex: Find a_n for $S_n = \dfrac{2n-1}{n+1}$?

sol/

$$S_n = S_{n-1} + a_n$$

$$S_{n-1} = \frac{2(n-1)-1}{(n-1)+1} = \frac{2n-2-1}{n} = \frac{2n-3}{n}$$

$$a_n = S_n - S_{n-1}$$

$$a_n = \frac{2n-1}{n+1} - \frac{2n-3}{n}$$

$$a_n = \frac{(2n-1)(n) - (2n-3)(n+1)}{(n+1)(n)} = \frac{(2n^2-n) - (2n^2+2n-3n-3)}{n^2+n}$$

$$a_n = \frac{\cancel{2n^2} - n - \cancel{2n^2} - 2n + 3n + 3}{n^2+n} = \frac{-n-2n+3n+3}{n^2+n} = \frac{\cancel{-3n}+\cancel{3n}+3}{n^2+n}$$

$$a_n = \frac{3}{n^2+n}$$

Two questions on Series

Find the Sum

- partial Sum (telescoping Serise)
- Geometric series

test for Conv. or div.

- divergent test
- P_test
- Integral test
- Compesion test
- limit comperssion test
- Ratio test
- Root test

1/ Partial Sum (telescoping test)

الشكل العام $\rightsquigarrow \sum_{n=1}^{\infty} f(n) - g(n)$

قانون الجمع $\rightsquigarrow S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

حالاته

- [1] directly (مباشر) ← اقتران - اقتران
- [2] Partial fraction (كسور جزئية) $\rightsquigarrow \sum \frac{1}{(f(n))(g(n))} = \frac{A}{f(n)} + \frac{B}{g(n)}$
- [3] خواص ln $\rightsquigarrow \sum \ln\left(\frac{A}{B}\right) = \sum \ln(A) - \ln(B)$

Ex Find the Sum for $\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+2}$?

Sol/ $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

$$S_n = \left(\boxed{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cdots$$

$$+ \left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}\right) + \left(\cancel{\frac{1}{n+1}} - \boxed{\frac{1}{n+2}}\right)$$

ملاحظة / نحسب القفزات بين دالتي الاقتران . نلاحظ هناك قفزة واحدة ، بالتالي نأخذ حد واحد في البداية وحد واحد في النهاية . فينتج ↓

$$S_n = \frac{1}{2} - \frac{1}{n+2}$$

نأخذ $\lim_{n\to\infty}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left\{\frac{1}{2} - \frac{1}{n+2}\right\} = \frac{1}{2} - \frac{1}{\infty}$$

$$= \frac{1}{2} - 0 = \boxed{\frac{1}{2}}$$

Ex: Find the sum of $\sum_{n=1}^{\infty} \left(11^{\frac{1}{n}} - 11^{\frac{1}{n+2}}\right)$?

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

$$S_n = \left(\boxed{11^{\frac{1}{1}}} - \cancel{11^{\frac{1}{3}}}\right) + \left(\boxed{11^{\frac{1}{2}}} - \cancel{11^{\frac{1}{4}}}\right) + \left(\cancel{11^{\frac{1}{3}}} - \cancel{11^{\frac{1}{5}}}\right) + \left(\cancel{11^{\frac{1}{4}}} - \cancel{11^{\frac{1}{6}}}\right) + \cdots$$

$$+ \left(\cancel{11^{\frac{1}{n-2}}} - \cancel{11^{\frac{1}{n}}}\right) + \left(\cancel{11^{\frac{1}{n-1}}} - \boxed{11^{\frac{1}{n+1}}}\right) + \left(\cancel{11^{\frac{1}{n}}} - \boxed{11^{\frac{1}{n+2}}}\right)$$

$$\therefore S_n = 11 + \sqrt{11} - 11^{\frac{1}{n+1}} - 11^{\frac{1}{n+2}}$$

$$= \lim_{n\to\infty} \left\{11 + \sqrt{11} - 11^{\frac{1}{n+1}} - 11^{\frac{1}{n+2}}\right\}$$

$$= 11 + \sqrt{11} - 11^{\frac{1}{\infty}} - 11^{\frac{1}{\infty}} \Rightarrow = 11 + \sqrt{11} - 11^0 - 11^0$$

$$= 11 + \sqrt{11} - 1 - 1 = 9 + \sqrt{11}$$

قفزتين يعني
أول حدين وآخر حدين

Ex: Find the sum of $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$?

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow$$

الحالة الثانية الكسور الجزئية

$$1 = A(n+1) + B(n) \rightsquigarrow n=0 \rightarrow 1 = A + 0 \Rightarrow A = 1$$

$$n=-1 \rightarrow 1 = 0 + -B \Rightarrow B = -1$$

نعيد كتابة السؤال
بتعويض A, B

$$\sum_{n=1} \frac{1}{n} - \frac{1}{n+1} \rightsquigarrow$$ اصبح اقتران

$$\therefore S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

$$S_n = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= \lim_{n\to\infty} \left\{1 - \frac{1}{n+1}\right\} = 1 - \frac{1}{\infty} = 1 - 0 = \boxed{1}$$

هناك قفزة واحدة
يعني أول وآخر حد فقط

Ex: Find the Sum (if exist) for $\sum_{n=1}^{\infty} \ln\left(\frac{n+3}{n+1}\right)$?

Sol/ $\sum_{n=1}^{\infty} \ln(n+3) - \ln(n+1)$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n$$

$$S_n = \left(\cancel{\ln(4)} - \boxed{\ln 2}\right) + \left(\cancel{\ln 5} - \boxed{\ln 3}\right) + \left(\ln 6 - \cancel{\ln 4}\right) + \left(\ln 7 - \cancel{\ln 5}\right) + \cdots$$
$$+ \left(\cancel{\ln(n+1)} - \cancel{\ln(n-1)}\right) + \left(\boxed{\ln(n+2)} - \cancel{\ln(n)}\right) + \left(\boxed{\ln(n+3)} - \cancel{\ln(n+1)}\right)$$

$$\therefore S_n = -\ln(2) - \ln(3) + \ln(n+2) + \ln(n+3)$$

$$\lim_{n\to\infty} \{-\ln(2) - \ln(3) + \ln(n+2) + \ln(n+3)\} = \boxed{\infty}$$

$$\checkmark \qquad \checkmark \qquad +\infty \qquad +\infty$$

نلاحظ في هذا السؤال أن الاختصار ليس على أول حد وآخر حد ...
وبالتالي يجب متابعة اختصار الحدود بالتوالي. لأن القفزة حادث
من الأكبر إلى الأصغر، وليس كما في الأمثلة السابقة كانت من الأصغر إلى الأكبر.

H.W/ Find Sum for $\sum_{n=1}^{\infty} \left(e^{\frac{4}{n}} - e^{\frac{4}{n+1}}\right)$

Ans: $e^4 - 1$

Geometric Series :-

The form of Series is :→

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

a : constant (ثابت)

r : اي عدد مرفوع للقوة (n)

$$\sum_{n=0}^{\infty} a(r)^n = \begin{cases} |r| \geq 1 \rightarrow \text{diverge} \rightarrow \boxed{\text{No sum}} \\ |r| < 1 \rightarrow \text{converge} \rightarrow \boxed{S_n = \frac{a_1}{1-r}} \end{cases}$$

a_1 / هو الحد الاول بعد تعويض n التي تبدأ بها المتسلسلة .

Ex: Find the Sum of the following Series :-

① $\sum_{n=0}^{\infty} \frac{2}{3^n}$

$$= \sum_{n=0}^{\infty} 2 \cdot \frac{1}{3^n} = \sum_{n=0}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^n \rightarrow r = \frac{1}{3} \quad , a_1 = 2\left(\frac{1}{3}\right)^0 = 2$$

$$S_n = \frac{a_1}{1-r} = \frac{2}{1-\frac{1}{3}}$$

$$S_n = \frac{2}{\frac{2}{3}} = 2 * \frac{3}{2} = \boxed{3}$$

$$r = \left|\frac{1}{3}\right| < 1 \rightarrow \text{converge}.$$

$\boxed{2}$ $\sum_{n=1}^{\infty} 2^{2n+1} \cdot 8^{-n}$

$$= \sum_{n=1}^{\infty} 2^{2n} \cdot 2^{1} \cdot \frac{1}{8^n} = \sum_{n=1}^{\infty} \frac{(2^2)^n \cdot 2}{8^n}$$

$$= \sum_{n=1}^{\infty} \frac{2(4)^n}{8^n} = \sum_{n=1}^{\infty} 2\left(\frac{4}{8}\right)^n$$

$$= \sum_{n=1}^{\infty} 2\left(\frac{1}{2}\right)^n$$

$\therefore a_1 = 1 \; , r = \frac{1}{2} < 1 \rightsquigarrow$ conv.

$$S_n = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{2}} = \boxed{2}$$

مهم $\boxed{3}$ $\sum_{n=1}^{\infty} \frac{3^{n-1} + 5^n}{4^{2n}}$ (نوزع البسط على المقام)

$$= \sum_{n=1}^{\infty} \frac{3^n \cdot 3^{-1}}{4^{2n}} + \sum_{n=1}^{\infty} \frac{5^n}{4^{2n}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{3} \cdot \frac{3^n}{(4^2)^n} + \sum_{n=1}^{\infty} \frac{5^n}{(4^2)^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{3}{16}\right)^n + \sum_{n=1}^{\infty} \left(\frac{5}{16}\right)^n$$

Note

conv ± conv = conv

conv ± div = div

div + div = div

div - div = !!

$$S_n = S_{n_1} + S_{n_2}$$

$$S_{n_1} = \frac{a_1}{1-r} \rightsquigarrow \left\{ \begin{array}{l} a_1 = \frac{1}{\cancel{3}}\left(\frac{\cancel{3}}{16}\right)^1 = \frac{1}{16} \\ r = \frac{3}{16} < 1 \rightsquigarrow \text{conv.} \end{array} \right\}$$

$$S_{n_2} = \frac{a_1}{1-r} \rightsquigarrow \left\{ \begin{array}{l} a_1 = 1\left(\frac{5}{16}\right)^1 = \frac{5}{16} \\ r = \frac{5}{16} < 1 \rightsquigarrow \text{conv.} \end{array} \right\}$$

$$S_n = \frac{\frac{1}{16}}{1-\frac{3}{16}} + \frac{\frac{5}{16}}{1-\frac{5}{16}} = \frac{1}{13} + \frac{5}{11} = \boxed{\frac{76}{143}}$$ - conv.

$\boxed{4}$ $$\sum_{n=1}^{\infty} \frac{3^n + 6^n}{5^{n+1}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{3^n}{5^n \cdot 5^1} + \frac{6^n}{5^n \cdot 5^1} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{5} \left(\frac{3}{5}\right)^n + \sum_{n=1}^{\infty} \frac{1}{5} \left(\frac{6}{5}\right)^n$$

$\Rightarrow S_n = \quad S_{n_1} + S_{n_2}$

For $S_{n_1} \rightsquigarrow \begin{cases} a_1 = \frac{1}{5}\left(\frac{3}{5}\right)^1 = \frac{3}{25} \\ r = \frac{3}{5} < 1 \rightsquigarrow \text{conv.} \end{cases}$

For $S_{n_2} \rightsquigarrow \begin{cases} a_1 = \frac{1}{5}\left(\frac{6}{5}\right)^1 = \frac{6}{25} \\ r = \frac{6}{5} > 1 \rightsquigarrow \text{div (have no sum)} \end{cases}$

$S_n = S_{n_1} + \text{No sum} \rightsquigarrow$ No sum

$= \text{conv.} + \text{div.} = \text{div}$