Lect. -8-

Sequences : المتتالية

المتتالية / هي مجموعة من القيم بينها فواصل تسير حسب قاعدة معينة تسمى بالقاعدة العامة . an . (وجميعها اعداد طبيعية) .

$$\{a_1 , a_2 , a_3 , \text{----------} a_n\} = \{a_n\}_{n=1}^{\infty}$$

Ex: write the first 3 terms of the Sequences ?

$\boxed{1}$ $a_n = \left\{\frac{2n}{n^2+1}\right\}_{n=1}^{\infty}$

Sol:

$$a_1 = \frac{2\times1}{(1)^2+1} = \frac{2}{2} = 1$$

$$a_2 = \frac{2\times2}{(2^2)+1} = \frac{4}{5}$$

$$a_3 = \frac{2\times3}{(3)^2+1} = \frac{6}{10} = \frac{3}{5}$$

$$\therefore \left\{1, \frac{4}{5}, \frac{3}{5}, \text{----}\right\}$$

$\boxed{2}$ $a_n = \left\{(-1)^n \sqrt{n-2}\right\}_2^6$

Sol/

$$a_2 = (-1)^2\sqrt{2-2} = 0$$

$$a_3 = (-1)^3\sqrt{3-2} = -1$$

$$a_4 = (-1)^4\sqrt{4-2} = \sqrt{2}$$

$$a_5 = (-1)^5\sqrt{5-2} = -\sqrt{3}$$

$$a_6 = (-1)^6\sqrt{6-2} = 2$$

$$\therefore a_n = \{0, -1, \sqrt{2}, -\sqrt{3}, 2\}$$

Ex: Find the nth term $\{a_n\}$ of the following sequences.

① $\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\}$

يمكن التعامل مع البسط بالبداية منفرداً للسهولة

$$a_n = \left\{\frac{n-1}{n}\right\}_{n=1}$$

② $\left\{0, \frac{1}{4}, \frac{2}{9}, \frac{3}{16}\right\}$

$$a_n = \left\{\frac{n-1}{n^2}\right\}_{n=1}$$

③ $\left\{2, 1, \frac{2^3}{3^2}, \frac{2^4}{4^2}, \frac{2^5}{5^2}\right\}$

$$a_n = \left\{\frac{2^n}{n^2}\right\}_{n=1}$$

④ $\left\{0, \frac{\ln 2}{2}, \frac{\ln 3}{3}, \frac{\ln 4}{4}\right\}$

$$a_n = \left\{\frac{\ln n}{n}\right\}_{n=1}$$

$\ln 1 = 0$

Ex: Find the formula of the general terms an for:-

1. $\{4, -1, \frac{1}{4}, \frac{-1}{16}, \frac{1}{64}, \ldots\}$

$n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5$

$4^{-1} \quad \frac{1}{4^0} \quad \frac{1}{4^1} \quad \frac{1}{4^2} \quad \frac{1}{4^3}$

$+ \quad - \quad + \quad - \quad +$

الخطوة الاولى/ نرقم العناصر بالتسلسل

الخطوة الثانية/ نلاحظ كيفية سير المتتالية

∴ هناك قيم سالبة وموجبة ∴ يجب ان يكون في الداله $(-1)^?$

البسط

نفرض $\rightarrow (-1)^n \rightarrow (-1)^1, (-1)^2, (-1)^3$

$-1 \quad 1 \quad -1$ ← الفرضية غير صحيحة

لان المتتالية تبدأ بموجب

نفرض $\rightarrow (-1)^{n+1} \rightarrow (-1)^{1+1}, (-1)^{2+1}, (-1)^{3+1}$

$+1, -1, +1$ ← ∴ الفرضية صحيحة

المقام

$\{4^{-1}, 4^0, 4^1, 4^2, 4^3\}$ ← يبدأ الأس من (-1)

∴ لايمكن ان نضع أس n أي (4^n) لأن n تبدأ من (1)

نفرض $\rightarrow (4)^{n-1} \rightarrow (4)^{1-1}, (4)^{2-1}, (4)^{3-1}$

$4^0, 4^1, 4^2\}$ ← لاتظهر (4^{-1}) الفرضية غير صحيحة

نفرض $\rightarrow (4)^{n-2} \rightarrow (4)^{1-2}, (4)^{2-2}, (4)^{3-2}, (4)^{4-2}$

$4^{-1}, 4^0, 4^1, 4^2$ ← الفرضية صحيحة

$$\therefore a_n = \left\{ \frac{(-1)^{n+1}}{(4)^{n-2}} \right\}_{n=1}^{\infty}$$

2 Find formula of

$$a_n = \left\{ 0, \frac{3}{5}, \frac{8}{10}, \frac{15}{17}, --- \right\} \ ?$$

$n=1 \quad n=2 \quad n=3 \quad n=4$

* نلاحظ سير المتتالية في البسط :-

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & \longrightarrow 3 & \longrightarrow 8 & \longrightarrow 15 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1^2-1 & 2^2-1 & 3^2-1 & 4^2-1 \end{array}$$

∴ البسط يتكون من $(n)^2-1$

* نلاحظ سير المتتالية في المقام :-

$$\begin{array}{ccc} 2 & 3 & 4 \\ 5 & \longrightarrow 10 & \longrightarrow 17 \\ \downarrow & \downarrow & \downarrow \\ (2)^2+1 & (3)^2+1 & (4)^2+1 \end{array}$$

∴ المقام يتكون من $(n)^2+1$

$$\therefore \quad a_n = \left\{ \frac{(n)^2-1}{(n)^2+1} \right\}$$

The limit of infinite Sequence:

$$\lim_{n\to\infty} a_n = \begin{cases} \lim\limits_{n\to\infty} a_n = \mp L & \text{(converge)} \\ \lim\limits_{n\to\infty} a_n = \mp\infty & \text{(diverge)} \\ \lim\limits_{n\to\infty} a_n = \begin{cases} L_1 \\ L_2 \end{cases} & \text{مختلفة (diverge)} \end{cases}$$

أي عدد

متشابهة (converge)

Convegence → تقارب

divergence → تباعد

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} = !!$$

→ C.R → $\ln$ عدم وجود → $\lim \dfrac{\text{اكبر أس بالبسط}}{\text{اكبر أس بالمقام}}$ = اختصار = ✓

→ L.R → $\ln$ بوجود → $\lim \dfrac{f'(x)}{g'(x)} = \text{———}$ ✓

اشتقاق (عدة مرات)

$\infty - \infty = !!$

→ توحيد مقام → اكثر من كسر

→ عامل مشترك → Polynomial

→ ضرب بالمرافق → بوجود الجذر $(\sqrt{\ } \mp \square)$

① $\lim\limits_{n\to\infty} \{a_n + b_n\} = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

② $\lim\limits_{n\to\infty} \{a_n \cdot b_n\} = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

③ $\lim\limits_{n\to\infty} \{k\, a_n\} = k \lim\limits_{n\to\infty} a_n$

④ $\lim\limits_{n\to\infty} \left\{\dfrac{a_n}{b_n}\right\} = \left\{\dfrac{\lim a_n}{\lim b_n}\right\}$, b_n is never 0

$b_n \neq 0$

خواص (∞)

1. $\infty + \infty = \infty$

2. $\infty * \infty = \infty$

3. عدد $* \infty = \mp \infty$

4. $\frac{\text{عدد}}{\infty} = 0$

5. $\frac{\infty}{\text{عدد}} = \mp \infty$

6. $\frac{\text{عدد}}{0} = \mp \infty$

7. $\infty \mp$ عدد $= \infty$

8. $\infty * 0 \longrightarrow$ يجب تغير شكل النهاية

- $AB = \frac{A}{\frac{1}{B}}$

- $e^{-n} = \frac{1}{e^{n}}$

9. $\ln(n+1)$
$= \ln(\infty + 1) = \infty$

$\left(\frac{\infty}{\infty}\right)$ and $(\infty - \infty) \longrightarrow$ طريقة حلها مذكورة في الصفحة السابقة

Ex: Determin the following Sequence Converge or diverge?

1) $a_n = \dfrac{3+5n^2}{n+n^2}$

$$\lim_{n\to\infty}\left\{\frac{3+5n^2}{n+n^2}\right\} = \frac{\infty}{\infty}!! \rightsquigarrow \text{C.R}$$

$$\therefore \lim_{n\to\infty} \frac{5n^2}{n^2} = \boxed{5} \to \therefore \text{Converge.}$$

2) $a_n = \dfrac{3n}{4+\sqrt{n}}$

$$\lim_{n\to\infty}\left\{\frac{3n}{4+\sqrt{n}}\right\} = \frac{\infty}{\infty}!! \longrightarrow \text{C.R}$$

$$\lim_{n\to\infty} \frac{3n}{\sqrt{n}} = \frac{3\cancel{\sqrt{n}}\sqrt{n}}{\cancel{\sqrt{n}}} = 3\sqrt{n} = \boxed{\infty} \longrightarrow \therefore \text{diverge.}$$

3) $\dfrac{4+3n^2}{1-n^3}$

$$\lim_{n\to\infty}\left\{\frac{4+3n^2}{1-n^3}\right\} = \frac{\infty}{-\infty}!! \longrightarrow \text{C.R}$$

$$\lim_{n\to\infty} \frac{3n^2}{-n^3} = \frac{-3}{n} = \frac{-3}{\infty} = \boxed{0} \longrightarrow \therefore \text{Converge.}$$

اي عدد مقسوم على $\infty = 0$.

4 $\lim_{n\to\infty} \{\frac{1}{n}\} = \frac{1}{\infty} = 0 \rightsquigarrow$ conv.

5 $\lim_{n\to\infty} \{n^2\} = \infty \rightsquigarrow$ diverge.

6 $\lim_{n\to\infty} \left\{\frac{3n^2-5n}{5n^2+2n+6}\right\} \rightsquigarrow \frac{\infty}{\infty}$!! C-R

$\lim_{n\to\infty} \left\{\frac{3n^2}{5n^2}\right\} = \frac{3}{5} \rightsquigarrow$ conv.

7 $\lim_{n\to\infty} \left\{\frac{3n^2-4n}{2n-1}\right\} = \frac{\infty}{\infty}$!! $\rightarrow$ C.R

$\lim_{n\to\infty} \left\{\frac{3n^2}{2n}\right\} = \frac{3n}{2} = \infty \rightsquigarrow$ diverge.

8 $\lim_{n\to\infty} \left\{\frac{2n-3}{3n-7}\right\}^4$

$\lim_{n\to\infty} \left\{\frac{2n}{3n}\right\}^4 = \lim_{n\to\infty} \left\{\frac{2}{3}\right\}^4 = \frac{16}{81} \rightsquigarrow$ conv.

9 $\lim_{n\to\infty} \left\{\frac{2^n \ln 2}{5}\right\} = \frac{\infty}{5} = \infty \longrightarrow$ diverge.

10 $\lim_{n\to\infty} \left\{\frac{\ln n}{e^n}\right\} \rightsquigarrow$ L.R (مشتقة) (L'Hopital's Rule

$\lim_{n\to\infty} \left\{\frac{1/n}{e^n}\right\} = \lim_{n\to\infty} \left\{\frac{1}{ne^n}\right\} = \frac{1}{\infty} = 0 \rightsquigarrow$ (converg)

Ex: Test conv. or div. ?

$\boxed{1}$ $a_n = \dfrac{(-1)^n \cdot n}{n+4}$

Sol/

زوجي

even $\longrightarrow \lim\limits_{n\to\infty} \dfrac{n}{n+4} = \dfrac{\infty}{\infty}\,!! \xrightarrow{C.R} = \lim\limits_{n\to\infty} \dfrac{n}{n} = \boxed{1}$

$\neq$ div.

فردي

odd $\longrightarrow \lim\limits_{n\to\infty} \dfrac{-n}{n+4} = \dfrac{-\infty}{\infty}\,!! \xrightarrow{C.R} = \lim\limits_{n\to\infty} \dfrac{-n}{n} = \boxed{-1}$

* اذا ظهرت قيم مختلفة يعني (diverge)

* 〃 〃 متشابهة (converge).

$\boxed{2}$ $a_n = 1 + (-1)^n$

$\lim\limits_{n\to\infty} \{ 1 + (-1)^n \}$

زوجي even $\longrightarrow \lim\limits_{n\to\infty} [1+1] = \boxed{2}$

فردي odd $\longrightarrow \lim\limits_{n\to\infty} [1-1] = \boxed{0}$ $\longrightarrow$ diverge

$\boxed{3}$ $a_n = \dfrac{\cos^n(\pi)}{n^2+3}$

$$= \frac{(\cos \pi)^n}{n^2+3} = \frac{(-1)^n}{n^2+3}$$

$$\text{even} \longrightarrow \lim_{n\to\infty} \frac{1}{n^2+3} = \frac{1}{\infty} = 0$$

$$\text{odd} \longrightarrow \lim_{n\to\infty} \frac{-1}{n^2+3} = \frac{-1}{\infty} = 0$$

→ (converge) قيم متشابهة

$\boxed{4}$ $a_n = \sqrt{n^2+n} - n$

$$\lim_{n\to\infty} \sqrt{n^2+n} - n = \infty - \infty !! \rightsquigarrow$$ {في حال وجود √ نضرب بالمرافق}

$$\therefore \lim_{n\to\infty} \left\{ \sqrt{n^2+n} - n * \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n} \right\} = \frac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}$$

$n^{\frac{2}{2}}$ $n^{\frac{1}{2}}$ n^1

$+n$ $+n$

$$= \frac{\infty}{\infty} !! \xrightarrow{C.R} \lim_{n\to\infty} \frac{n}{2n}$$

$$= \boxed{\frac{1}{2}} \longrightarrow \text{convege.}$$

في حال وجود أكثر من حد لهما نفس الأس (نأخذ الاثنين)

$\boxed{5}$ $a_n = \sqrt{n+1} - \sqrt{n}$

$$= \lim_{n\to\infty} \{\sqrt{n+1} - \sqrt{n}\} = \infty - \infty = !! \rightsquigarrow$$ {ضرب بالمرافق}

$$= \lim_{n\to\infty} \left\{ (\sqrt{n+1} - \sqrt{n}) * \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \right\} = \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} =$$

$$= \lim_{n\to\infty} \left\{ \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\infty+\infty} = \frac{1}{\infty} = \boxed{0} \rightsquigarrow \text{converg}\right.$$

Monotone Sequences المتتابعات الرتيبه

DEFINITION A sequence $\{a_n\}_{n=1}^{+\infty}$ is called

1- **increasing** *(strictly increasing)* if $a_1 < a_2 < a_3 < \cdots a_n < a_{n+1}$ for all $n \geqslant 1$,

Ex: $\{4,6,8,10,....\}$ $4 < 6 < 8 < 10$ متزايده تماما

2- **decreasing** *(strictly decreasing)* if $a_1 > a_2 > a_3 > .... a_n > a_{n+1}$ for all $n \geqslant 1$.

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} \cdots$ $\{-1, -2, -3, -4, \ldots\}$ $-1 > -2 > -3 > -4 \cdots$

3- Non decreasing if $a_1 \leq a_2 \leq a_3 \leq \ldots \boxed{a_n \leq a_{n+1}}$ $\{2,2,4,4,6,6,..\}$

ليست تناقصيه يعني ممكن تكون تزايديه وممكن تكون ثابته

$a_n \leq a_{n+1} \Rightarrow$ ممكن كتابتها $\boxed{a_{n+1} - a_n \geqslant 0}$ وممكن كتابتها بالقسمه على a_n $\boxed{\frac{a_{n+1}}{a_n} \geqslant 1}$

4- Non increasing if $a_1 \geqslant a_2 \geqslant a_3 \geqslant .... a_n \geqslant a_{n+1}$ $\{-2,-2,-4,-4,-6,-6 ,\cdots\}$

ليست تزايديه يعني ممكن تكون تناقصيه وممكن تكون ثابته

$a_n \geqslant a_{n+1} \Rightarrow \boxed{a_{n+1} - a_n \leq 0} \Rightarrow \boxed{\frac{a_{n+1}}{a_n} \leq 1}$

A sequence is **monotonic** if it is either increasing or decreasing.

the sequence $a_n = (-1)^n$ is $\{-1, 1, -1, 1, -1, 1, -1, \ldots\}$

not increasing ,not decreasing ..not monotonic seq.

Examples:

1- $\{\frac{1}{n^2}\} = \{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\}$ ***(Decreasing)***

2- $\{\frac{n-1}{n}\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$ increasing

3- $\{(-1)^{n+1}\} = \{1, -1, 1, -1, 1, -1, \ldots\}$ *(not monotonic seq .)*

To show that the seqence $\{a_n\}$ Strictly increasing or strictly decreasing it is enough to show that:

sequence	Difference	Ratio(term $\pm$)	Derivative
increasing	$a_{n+1} - a_n > 0$	$\frac{a_{n+1}}{a_n} > 1$	$f'(x) > 0$
decreasing	$a_{n+1} - a_n < 0$	$\frac{a_{n+1}}{a_n} < 1$	$f'(x) < 0$

To show that the seqence $\{a_n\}$ is **Non** decreasing *or* **Non** increasing *it is enough to show that:*

sequence	Difference	Ratio(term $+$)	Derivative
Non decreasing	$a_{n+1}-a_n \geqslant 0$	$\frac{a_{n+1}}{a_n} \geqslant 1$	$f'(x) \geqslant 0$
Non increasing	$a_{n+1}-a_n \leq 0$	$\frac{a_{n+1}}{a_n} \leq 1$	$f'(x) \leq 0$

Ex: Use three different methods , to show that $\left\{\frac{n+1}{n+3}\right\}_{n=1}^{\infty}$ ***is strictly increasing .***

Method 1: (Difference)

$$a_{n+1}=\frac{(n+1)+1}{(n+1)+3}=\frac{(n+2)}{(n+4)} \quad , \quad a_n=\frac{n+1}{n+3}$$

$$a_{n+1}-a_n=\frac{(n+2)}{(n+4)}-\frac{n+1}{n+3}$$

$$=\frac{(n+2)(n+3)-(n+4)(n+1)}{(n+4)(n+3)}$$

$$=\frac{n^2+5n+6-(n^2+5n+4)}{n^2+7n+12}$$

$$a_{n+1}-a_n=\frac{2}{n^2+7n+12}>0 \quad \text{for all } n=1,2,3,\ldots$$

$\therefore \left\{\frac{n+1}{n+3}\right\}_{n=1}^{\infty}$ ***is strictly increasing .***

Method 2 : Ratio method)

$$a_{n+1}=\frac{(n+1)+1}{(n+1)+3}=\frac{(n+2)}{(n+4)} \quad , \quad a_n=\frac{n+1}{n+3}$$

$$\frac{a_{n+1}}{a_n}=\frac{\frac{(n+2)}{(n+4)}}{\frac{n+1}{n+3}}=\frac{(n+2)}{(n+4)}\cdot\frac{n+3}{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^2+5n+6}{n^2+5n+4}$$

$$= \frac{(n^2+5n+4)+2}{(n^2+5n+4)} > 1 \text{ for all } n = 1,2,3,....$$

$\therefore \left\{\frac{n+1}{n+3}\right\}_{n=1}^{\infty}$ ***is strictly increasing .***

Method 3 : (Derivative) $\left\{\frac{n+1}{n+3}\right\}_{n=1}^{\infty}$

$$a_n = \frac{n+1}{n+3}$$

Let $f(x) = \frac{x+1}{x+3}$, $x \in [1, \infty)$

$$f'(x) = \frac{x+3-(x+1)}{(x+3)^2}$$

$$= \frac{2}{(x+3)^2} > 0 \text{ for all } n = 1,2,3,....$$

$\therefore$ f ***is strictly increasing .***

$\therefore \left\{\frac{n+1}{n+3}\right\}_{n=1}^{\infty}$ ***is strictly increasing .***