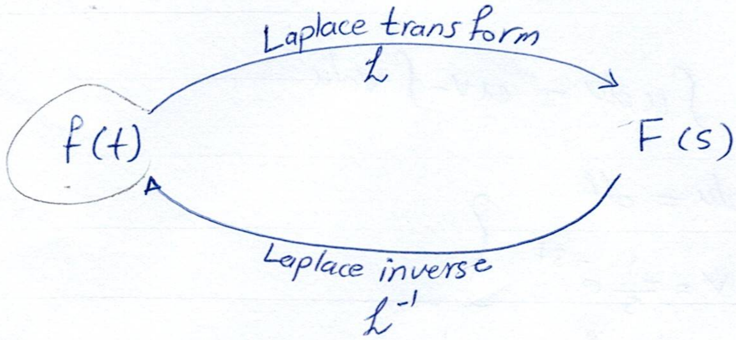
Lecture -

Laplace Transform



$$\boxed{F(s) = \int_0^\infty f(t) \cdot e^{-st} \cdot dt} \rightarrow s > 0$$

Ex: Find $F(s)$ For $f(t) = 1$, $t > 0$.

$$F(s) = \int_0^\infty f(t) \cdot e^{-st} \cdot dt$$

$$= \int_0^\infty 1 \cdot e^{-st} \cdot dt = \left[\frac{-1}{s} e^{-st} \right]_0^\infty$$

$$= \left[\frac{-1}{s} e^{-\infty} + \frac{1}{s} e^0 \right] = 0 + \frac{1}{s} = \boxed{\frac{1}{s}}$$

Ex: Find $F(s)$ For $f(t) = e^{at}$.

$$F(s) = \int_0^\infty f(t) \cdot e^{-st} \cdot dt$$

$$= \int_0^\infty e^{at} \cdot e^{-st} \cdot dt = \left[\int_0^\infty e^{-(s-a)t} \cdot dt \right]$$

$$= \left[\frac{1}{-(s-a)} e^{-(s-a)t} \right]_0^\infty = \frac{1}{-(s-a)} e^{-\infty} - \frac{1}{-(s-a)} e^0$$

$$= 0 + \frac{1}{s-a} \Rightarrow \mathcal{L}(e^{at}) = \frac{1}{s-a}$$

Ex: $f(t)=t$, Find $F(s)$?

$$F(s) = \mathcal{L}(t) = \int_0^\infty f(t) \cdot e^{-st} \cdot dt$$

$$F(s) = \int_0^\infty t \cdot e^{-st} \cdot dt$$

by using $\int u\,dv = uv - \int v\,du$

$$\left\{ \begin{array}{l} u = t \Rightarrow du = dt \\ dv = e^{-st} \Rightarrow v = \frac{-1}{s} e^{-st} \end{array} \right\}$$

$$\therefore F(s) = \frac{-t}{s} e^{-st} + \int \frac{1}{s} e^{-st} \cdot dt$$

$$= \left[\frac{-t}{s} e^{-st} + \frac{1}{s(-s)} e^{-st} \right]_0^\infty$$

$$= \left(\frac{-t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right) \Big|_0^\infty$$

$$= (0-0) - \left(0 - \frac{1}{s^2}\right) = \frac{1}{s^2}$$

$$\therefore \boxed{\mathcal{L}(t) = \frac{1}{s^2}}$$

Lecture -[illegible]-

Laplace Transform

Ex: if $f(t) = t^2$, find $\mathcal{L}(f(t))$. ?

Sol: $F(s) = \mathcal{L}(f) = \int_0^\infty f(t)\, e^{-st} . dt$

$= \int_0^\infty t^2 . e^{-st} . dt$

by using $\int u\, dv = uv - \int v . du$

$\therefore \left\{ \begin{array}{ll} u = t^2 , & du = 2t . dt \\ dv = e^{-st} , & v = -\frac{1}{s} e^{-st} \end{array} \right\}$

$\therefore F(s) = \frac{-t^2}{s} e^{-st} - \int -\frac{1}{s} e^{-st} . 2t . dt$

$= \frac{-t^2}{s} e^{-st} + \frac{2}{s} \int t . e^{-st} . dt$

again wehaveto use another integral.

$u = t \quad , du = dt$

$dv = e^{-st} \quad , \quad v = \frac{-1}{s} e^{-st}$

$F(s) = \frac{-t^2}{s} e^{-st} + \frac{2}{s} \left[-\frac{t}{s} e^{-st} + \int \frac{1}{s} e^{-st} . dt \right]$

$F(s) = \frac{-t^2}{s} e^{-st} + \frac{2}{s} \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right] \Big|_0^\infty$

$= \Big| \frac{-t^2}{s} e^{-st} - \frac{2t}{s^2} e^{-st} - \frac{2}{s^3} e^{-st} \Big|_0^\infty$

$= 0 \qquad 0 \qquad \cancel{\frac{-2}{s^3} e^{-\infty}} - \left(\frac{-2}{s^3} e^{0} \right)$

0

$\therefore \boxed{F(s) = \frac{2}{s^3}} \Rightarrow \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$$\therefore \boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$$ $\to$ at n: عدد صحيح موجب

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* The Gamma function : $\Gamma(n)$:-

Gamma function has the following properties :-

1) $\Gamma(n+1) = n!$ $\to$ n: عدد صحيح موجب
2) $\Gamma(n+1) = n\,\Gamma(n)$ $\to$ n: كسر موجب
3) $\Gamma(n) = \frac{\Gamma(n+1)}{n}$ $\to$ n: كسر سالب
4) $\Gamma(-n) = \infty$ $\to$ n: عدد صحيح سالب
5) $\Gamma(0) = +\infty$ $\to$ n: zero
6) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Examples :

① $\Gamma(3) = n! = 2! = 2*1 = 2$

② $\Gamma(5) = 4! = 4*3*2*1 = 24$

③ $\Gamma(1) = 0! = 1$

④ $\Gamma(\frac{3}{2}) \Rightarrow n+1 = \frac{3}{2}$, $n = \frac{3}{2} - 1$, $n = \frac{1}{2}$ $\therefore \Gamma(\frac{3}{2}) = n\,\Gamma(n)$

$\Rightarrow = \frac{1}{2}\,\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$.

⑤ $\Gamma(\frac{7}{2}) \Rightarrow n+1 = \frac{7}{2}$, $n = \frac{7}{2} - 1$, $n = \frac{5}{2}$ $\therefore \Gamma(\frac{7}{2}) = \frac{5}{2}\,\Gamma(\frac{5}{2})$

$\Rightarrow = \frac{5}{2} \cdot ((\frac{5}{2} - 1) \cdot \Gamma(\frac{3}{2}) \Rightarrow = \frac{5}{2} * \frac{3}{2} * (\frac{3}{2} - 1)\,\Gamma(\frac{1}{2})$

$\Rightarrow = \frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$

⑥ $\Gamma(\frac{-1}{2}) \Rightarrow n = \frac{-1}{2}$, $\Gamma(n) = \frac{\Gamma(n+1)}{n}$, $\Gamma(\frac{-1}{2}) = \frac{\Gamma(-\frac{1}{2}+1)}{-\frac{1}{2}}$

$\Rightarrow = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\sqrt{\pi}$

⑦ $\Gamma(-1) = -\infty$

* The Laplace tansform of t^n can be defined by Gamma function

$$\boxed{\mathcal{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}}$$

* If n is an integer $\Rightarrow \Gamma(n+1) = n!$, and

$$\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$$

Example: Find Laplace transform.

① e^{-3t} : $\mathcal{L}(e^{-3t}) = \frac{1}{s+3}$

Some functions f(t) and their Laplace transform $\mathcal{L}$ (f(t)) .

f(t)	F(s)
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$, n: عدد صحيح موجب
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
e^{-at}	$\frac{1}{s+a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$

Ex: Find $\mathcal{L}(t)$ of:

① $\mathcal{L} e^{-3t} = \frac{1}{s+3}$

② $\mathcal{L} \cos 5t = \frac{s}{s^2+25}$

③ $\mathcal{L} \sinh 4t = \frac{4}{s^2-16}$

④ $\mathcal{L}(t^3) = \frac{n!}{s^{n+1}} = \frac{3*2*1}{s^{3+1}} = \frac{6}{s^4}$

⑤ $\mathcal{L}(t^{5/2}) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{\Gamma(\frac{7}{2})}{s^{7/2}} = \frac{\frac{5}{2} * \frac{3}{2} * \frac{1}{2} \Gamma(\frac{1}{2})}{s^{\frac{7}{2}}} = \frac{15}{8} \frac{\sqrt{\pi}}{s^{7/2}}$

⑥ $\mathcal{L}(t^{-\frac{1}{2}}) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$

⑦ $\mathcal{L}(t^1) = \frac{n!}{s^{n+1}} = \frac{1}{s^2}$

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* Properties of Laplace transform.

① Linearity :-

Ex: $\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\}$

$$= 4 \cdot \frac{2!}{s^3} - 3\frac{s}{s^2+4} + 5\frac{1}{s+1}$$

$$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$$

② Time shifting :-

If $\mathcal{L} f(t) = F(s)$

* $\mathcal{L} f(t \mp a) = e^{\mp as} \cdot F(s)$

Ex: $\mathcal{L}\{(t-6)^3\} = e^{-6s} \cdot \frac{3!}{s^4} = \frac{6e^{-6s}}{s^4}$

③ S_Shifting (multiplication f(t) by e^{at})

$$\mathcal{L}\{f(t).e^{\mp at}\} = F(s \mp a)$$

(Replacing s by $(s \mp a)$ in the transform

Ex: $\mathcal{L}\, e^{-2t} \cos 4t$

Sol: $\mathcal{L} \cos 4t = \frac{s}{s^2+16}$

then, $\mathcal{L}\, e^{-2t} \cos 4t = \frac{(s+2)}{(s+2)^2+16} = \frac{(s+2)}{s^2+4s+20}$

④ L.T of derivatives: (بالإشتقاق)

If $\mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\{f'(t)\} = s.F(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - s f(0) - f'(0)$$

Ex: find $\mathcal{L}\{f'(t)\}$ for $f(t) = \cos(3t)$

Sol: (2 way)

* method one:-

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+9}$$

$$\mathcal{L}\{f'(t)\} = s.\frac{s}{s^2+9} - 1$$

$f(0) = \cos 3(0) = 1$

$$= \frac{s^2}{s^2+9} - 1 = \frac{\cancel{s^2} - \cancel{s^2} - 9}{s^2+9} =$$

$$= \boxed{\frac{-9}{s^2+9}}$$

* method two:

$f(t) = \cos 3t$, $f'(t) = -3 \sin 3t$

$$\mathcal{L}\{f'(t)\} = -3\frac{3}{s^2+9} = \boxed{\frac{-9}{s^2+9}}$$

(5) L.T of Integral : (تحويل التكامل)

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

* Integration in time domire ⇔ division by (s) in s-domine.

Ex: Find $\mathcal{L}\left\{\int_0^t \sin 2\tau\,d\tau\right\}$

Sol ⇒

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4} \quad \text{, then}$$

$$\mathcal{L}\left\{\int_0^t \sin 2\tau\,d\tau\right\} = \left(\frac{2}{s^2+4}\right)/s$$

$$= \frac{2}{s(s^2+4)}$$

Ex: Find $\mathcal{L}\left\{\int_0^t (\tau^2 - \sinh 3\tau + 5e^{-3\tau})\,d\tau\right.$

Sol

$$= \frac{1}{s}\left(\frac{2}{s^3} - \frac{3}{s^2-9} + 5\frac{1}{s+3}\right)$$

(6) Multiplication by (t^n) :

$$\mathcal{L}\{t.f(t)\} = -\frac{d}{ds}F(s)$$

in general

$$\mathcal{L}\{t^n.f(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

* Multiplication in t-domine ⇔ defferentiation in s-domine

Ex: Find L.T of ① te^{2t} ?

Sol:

$$\mathcal{L}\{t.f(t)\} = -\frac{d}{ds}F(s)$$

$$= -\frac{d}{ds}\left(\frac{1}{s-2}\right)$$

$$= -\left[\frac{(s-2)(0)-(1)(1)}{(s-2)^2}\right] = \frac{1}{(s-2)^2}$$

② $t^2 e^{2t}$

Sol: $\mathcal{L}\{t^n . f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

$$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right)$$ (المشتقة الثانية)

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s-2}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{1}{(s-2)^2}\right]$$

$$= \frac{\cancel{(s-2)^2(0)} - (1)(2(s-2))}{(s-2)^4}$$

$$= \frac{-2\cancel{(s-2)}}{(s-2)^{\cancel{4}}} = \frac{-2}{(s-2)^3}$$

Ex: Find L.T of $f(t) = t e^{-2t} \sinh 3t$?

Sol:

$\sinh 3t \xrightarrow{\mathcal{L}} \frac{3}{s^2-9}$ الحل يكون على مراحل لتبسيط المسألة كما يلي :-

$$\mathcal{L}(t \sinh 3t) = \frac{-d}{ds} \cdot \left(\frac{3}{s^2-9}\right) = \frac{-(3)(2s)}{(s^2-9)^2} = \frac{6s}{(s^2-9)^2}$$

then, using property of (s-shifting) multiplies by (e^{at})

$$\mathcal{L}(t \sinh 3t \cdot e^{-2t}) = F(s \mp a)$$

$$= \frac{6(s+2)}{((s+2)^2 - 9)^2}$$

* في هذا المثال تم استخدام أكثر من خاصية في آن واحد.

7 Division by t :-

$$\mathcal{L}\left\{ \frac{f(t)}{t} \right\} = \int_{s}^{\infty} F(u)\,du$$

* Division by t in t-domien ⇔ Integration in s-domine

Ex: Find L.T of $\left(\frac{\sin t}{t}\right)$?

Sol:

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1}$$

$$\mathcal{L}\left(\frac{\sin t}{t}\right) = \int_{s}^{\infty} \frac{1}{u^2+1}\,du$$

$$= \tan^{-1} u \Big|_{s}^{\infty} = \tan^{-1}(\infty) - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

Ex: Find L.T of $\left(\frac{e^{2t}-1}{t}\right)$?

Sol:

$$\mathcal{L}(e^{2t}-1) = \frac{1}{s-2} - \frac{1}{s}$$

$$\mathcal{L}\left(\frac{e^{2t}-1}{t}\right) = \int_{s}^{\infty} \left(\frac{1}{u-2} - \frac{1}{u}\right) du$$

$$= \ln(u-2) \Big|_{s}^{\infty} - \ln u \Big|_{s}^{\infty}$$

$$= \cancel{\infty} - \ln(s-2) - \cancel{\infty} + \ln s$$

$$= \ln s - \ln(s-2)$$

$$= \ln \frac{s}{s-2}$$

Ex: Find F(s) For the following functions:-

① $f(t) = e^{2t}\, u(t-2)$

* تطبق خاصية (s-shifting) على هذا السؤال. لكون الدالة مضروبة بـ (e^{at})

الخطوة الأولى/

$$\mathcal{L}\{u(t-2)\} = e^{-2s}\cdot\frac{1}{s} = \frac{e^{-2s}}{s}$$

الخطوة الثانية/ $\mathcal{L}\{e^{2t}\,u(t-2)\} = \dfrac{e^{-2(s-2)}}{(s-2)}$

② $f(t) = t.e^{-3t}.\sin 2t$

* أكثر من خاصية بنفس السؤال
* نبدأ بالدالة التي ليس لها تأثير على بقية الدوال (sin)

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{t\sin 2t\} = \frac{-d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -\left[\frac{(s^2+4)(0)-(2)(2s)}{(s^2+4)^2}\right] = \frac{4s}{(s^2+4)^2}$$

$$\mathcal{L}\{t\,e^{-3t}\sin 2t\} = \frac{4(s+3)}{((s+3)^2+4)^2} = \frac{4s+12}{(s^2+6s+13)^2}$$

③ $f(t) = \dfrac{1-\cos 3t}{t}$

$$\mathcal{L}\{1-\cos 3t\} = \frac{1}{s} - \frac{s}{s^2+9}$$

$$\mathcal{L}\left\{\frac{1-\cos 3t}{t}\right\} = \int_s^\infty \left[\frac{1}{u} - \frac{u}{u^2+9}\right]\cdot du$$

$$= \left[\ln u - \frac{1}{2}\ln(u^2+9)\right]_s^\infty \Rightarrow = \left[(\cancel{\infty} - \ln s) - \frac{1}{2}(\cancel{\infty} - \ln(s^2+9))\right]$$

$$= -\ln s + \frac{1}{2}\ln(s^2+9) \Rightarrow = \ln(s^2+9)^{\frac{1}{2}} - \ln s$$

$$= \ln\frac{\sqrt{s^2+9}}{s}$$

(4) $f(t) = e^{-3t} \int_0^t \frac{\sin t}{t} . dt$

$$\mathcal{L}\left\{ \frac{\sin t}{t} \right\} = \int_s^\infty \frac{1}{u^2+1} . du$$

$$= \left[\tan^{-1} u \right]_s^\infty = \left(\tan^{-1} \infty - \tan^{-1} s \right]$$

$$= \left(\frac{\pi}{2} - \tan^{-1} s \right)$$

$$\frac{\pi}{2} - \tan^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$$

$$= \tan^{-1}\left(\frac{1}{s}\right)$$

$$\mathcal{L}\left\{ \int_0^t \frac{\sin t}{t} . dt \right\} = \left(\tan^{-1}\left(\tfrac{1}{s}\right)\right) / s = \frac{1}{s} \tan^{-1}\left(\frac{1}{s}\right)$$

$$\mathcal{L}\left\{ e^{-3t} \int_0^t \frac{\sin t}{t} . dt \right\} = \frac{1}{(s+3)} \tan^{-1}\left(\frac{1}{s+3}\right)$$

Inverse Laplace Transform $.(\mathcal{L}^{-1})$

$$\boxed{f(t) = \mathcal{L}^{-1}\{F(s)\}}$$

Methods of finding $\mathcal{L}^{-1}$:-

① by using the table of laplace tansform.

Ex:- Find the inverse tansform laplace of the functions:-

ⓐ $F(s) = \frac{3}{s}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} = 3\,\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 3\ u(t)$

b) $F(s) = \frac{1}{s+3}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$

ⓒ $F(s) = \frac{1}{s^2+9}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) = \frac{1}{3}\sin 3t$

ⓓ $F(s) = \frac{1}{s^4}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}\mathcal{L}^{-1}\left(\frac{3!}{s^4}\right) = \frac{1}{6}t^3$

c) $F(s) = \frac{3s}{s^2-25}$

$f(t) = \mathcal{L}^{-1}\left(\frac{3s}{s^2+25}\right) = 3\cosh 5t$

② by using the Properties of Laplace transform.

Ex: Find $\mathcal{L}^{-1}$ of the function:-

① $F(s) = \dfrac{4}{s-2} - \dfrac{3s}{s^2+16} + \dfrac{4}{s^2+4}$

Sol/ $\mathcal{L}^{-1}\dfrac{4}{s-2} - \mathcal{L}^{-1}\dfrac{3s}{s^2+16} + \mathcal{L}^{-1}\dfrac{4}{s^2+4}$ (linearity)

$4e^{2t} - 3\cos 4t + 2\sin 2t$

② $F(s) = \dfrac{1}{(s-2)^2+4}$ (using s-shifting)

$\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\} = \dfrac{1}{2}\sin 2t$

$\mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)^2+4}\right\} = \dfrac{1}{2}e^{2t}\sin 2t$

③ $F(s) = \dfrac{e^{-2s}}{s^2+1}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$ (time shifting)

$\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2+1}\right\} = \sin(t-2)$

④ property of $\mathcal{L}^{-1}$ of derivative is:

(ln, $\tan^{-1}$)

$$\mathcal{L}^{-1}\{F^{n}(s)\} = \mathcal{L}^{-1}\left\{\frac{d^n F(s)}{ds^n}\right\} = (-1)^n t^n . f(t)$$

Ex: $\mathcal{L}^{-1}\{\ln(n+2)\} - \mathcal{L}^{-1}\left\{\dfrac{d}{ds}\ln(s+2)\right\} = -t\, f(t)$

$= \mathcal{L}^{-1}\left(\dfrac{1}{s+2}\right) = -t\, f(t)$

$= e^{-2t} = -t\, f(t) \Rightarrow f(t) = \dfrac{-e^{-2t}}{t}$

Q/ $F(s) = \ln\left(\dfrac{s+3}{s+4}\right) \rightarrow f(t) = ?$

$f(t) = \dfrac{e^{-3t} - e^{-4t}}{-t}$

⑤ Applying rule of division by s property (Integral)

Ex:/ $F(s) = \dfrac{1}{s(s+1)}$

$$\mathcal{L}^{-1} \frac{F(s)}{s} = \int_0^t f(\tau)\,d\tau$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \left\{\mathcal{L}^{-1} \frac{1}{s} \cdot \frac{1}{s+2}\right\} = \int_0^t e^{-2\tau}\,d\tau$$

$$= \left[\frac{-1}{2} e^{-2\tau}\right]_0^t = \frac{-1}{2}\left(e^{-2t} - e^{0}\right)$$

$$= \frac{1 - e^{-2t}}{2}$$

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⑥ Applying multiplication by s property :- (derivative)

$$\mathcal{L}^{-1}\{s F(s)\} = f'(t) + f(0)$$

Ex/ $F(s) = \dfrac{s^2}{s^2+4}$

$$\mathcal{L}^{-1}\left\{s \cdot \frac{s}{s^2+4}\right\} = \frac{d}{dt}(\cos 2t) + \cos(0)$$

$$= -2\sin 2t + 1$$

Ex: $F(s) = \dfrac{1}{4s+5}$, find $\mathcal{L}^{-1}$

$$= \frac{1}{4\left(s+\frac{5}{4}\right)}$$

$$= \frac{1}{4} \mathcal{L}^{-1} \frac{1}{\left(s+\frac{5}{4}\right)} = \frac{1}{4} e^{-5/4}$$

« Partial Fraction method »

if $F(s) = \frac{G(s)}{H(s)}$; where $G(s)$ and $H(s)$ are a polynomials in (s) and the degree of $G(s)$ is lower than of $H(s)$, then $F(s)$ can be fractioned.

case 1: (Simple Poles)

if $H(s)$ is in form of (unrepeated factors)

$$F(s) = \frac{A}{(S-a)} + \frac{B}{(S-b)} + \frac{C}{(S-c)} + \cdots$$

then, find A, B, C and L^{-1} for each function.

Ex:

find $L^{-1}\left\{\frac{3S+7}{S^2-2S-3}\right\}$

$$\frac{3S+7}{S^2-2S-3} = \frac{3S+7}{(S-3)(S+1)} = \frac{A}{S-3} + \frac{B}{S+1}$$

A: ضرب المعادلة في مقام A

$$\left[(\cancel{S-3}) * \frac{3S+7}{(\cancel{S-3})(S+1)}\right]_{S=3} = \frac{3S+7}{S+1} = \frac{9+7}{3+1} = \frac{16}{4} = \boxed{4}$$

B: ضرب المعادلة في مقام B

$$\left[(\cancel{S+1}) \cdot \frac{3S+7}{(S-3)(\cancel{S+1})}\right]_{S=-1} = \frac{3S+7}{S-3} = \frac{-3+7}{-1-3} = \frac{4}{-4} = \boxed{-1}$$

$$\therefore L^{-1}\left\{\frac{3S+7}{S^2-2S-3}\right\} = L^{-1}\left\{\frac{4}{S-3} + \frac{-1}{S+1}\right\}$$

$$= 4e^{3t} - e^{-t}$$

$$\frac{3S+7}{(S-3)(S+1)} = \frac{A}{S-3} + \frac{B}{S+1}$$ طريقة حل آخرى :-

$$= \frac{A(S+1) + B(S-3)}{(S-3)(S+1)}$$

$$\therefore (3S+7) = AS + A + BS - 3B$$

$$3S+7 = (A+B)S + (A-3B)$$ → تساوي الحدود المتشابهة

* $A+B = 3$ ①
* $A-3B = 7$ ②

$A = 4 \quad , B = -1$

$$\therefore \mathcal{L}^{-1}\left\{\frac{3S+7}{S^2-2S+3}\right\} = 4e^{3t} - e^{-t}$$

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case 2: (Multiple Poles)

if H(s) is in the form of Polynomial raised to (Some Power) $(S-a)^m$, (repeated factor.)

$$F(s) = \frac{A}{(S-a)^m} + \frac{B}{(S-a)^{m-1}} + \text{------} + \frac{K}{(S-m)}$$

finding A, B, C, ----

$$A_K = \frac{1}{(m-n)!} \cdot \frac{d^{m-n}}{ds^{m-n}} Q(s)$$

where $Q(s) = \dfrac{(S-a)^m \; G(s)}{H(s)}$

Ex: find $\mathcal{L}^{-1}$ of $F(s) = \dfrac{10-4s}{(s-2)^4}$

$$F(s) = \frac{A}{(s-2)^4} + \frac{B}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)}$$

$$A = \frac{10-4s}{\cancel{(s-2)^4}} \cdot \cancel{(s-2)^4}\Big]_{s=2} = 10-4(2) = \boxed{2}$$

$$B = \frac{1}{(4-3)!}\left[\frac{d}{ds}\frac{(10-4s)}{1}\right]_{s=2} = \frac{1}{1!}(-4) = \boxed{-4}$$

$$C = \frac{1}{(4-2)!}\left[\frac{d^2}{ds^2}(10-4s)\right]_{s=2} = \frac{1}{2!}\left[\frac{d}{ds}(-4)\right] = \frac{1}{2}(0) = \boxed{0}$$

$$D = 0$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{10-4s}{(s-4)^2}\right\} = \frac{2}{(s-2)^4} - \frac{4}{(s-2)^3} + 0 + 0$$

$$= 2\,\mathcal{L}^{-1}\frac{1}{(s-2)^4} - 4\,\mathcal{L}^{-1}\frac{1}{(s-2)^3}$$

$$= 2 \cdot e^{2t} \cdot \frac{t^3}{3!} - 4\,e^{2t} \cdot \frac{t^2}{2!}$$

$$= \frac{2}{6}t^3 e^{2t} - 2t^2 e^{2t}$$

$$\therefore f(t) = \mathcal{L}^{-1} = e^{2t}\left(\frac{1}{3}t^3 - 2t^2\right)$$

Q/ $F(s) = \dfrac{5s^2 - 15s - 11}{(s+1)(s-2)^3}$ $\rightarrow$

Q/ $F(s) = \dfrac{s^2 - 5s + 7}{(s+2)^3}$

Case 3: Quadratic factor (s^2)

Ex: Find $\mathcal{L}^{-1}\left\{\dfrac{3S+1}{(S-1)(S^2+1)}\right\}$

Sol/ $$\frac{3S+1}{(S-1)(S^2+1)} = \frac{A}{(S-1)} + \frac{BS+C}{(S^2+1)}$$

$$A = \left[\cancel{(S-1)} * \frac{3S+1}{(S^2+1)\cancel{(S-1)}}\right]_{S=1} = \frac{3(1)+1}{(1)^2+1} = \boxed{2}$$

in case of B and C we follow the method below :

$$\frac{3S+1}{(S-1)(S^2+1)} = \frac{A(S^2+1)+(BS+C)(S-1)}{(S-1)(S^2+1)}$$ (طريقة ضرب الوسطين × الطرفين)

$$3S+1 = AS^2+A+BS^2-BS+CS-C$$

$$3S+1 = (A+B)S^2+(-B+C)S+A-C$$

$$\therefore \begin{bmatrix} A+B=0 \\ -B+C=3 \\ A-C=1 \end{bmatrix}$$

we find $\boxed{A=2}$

So, $A=-B \longrightarrow \boxed{B=-2}$

$A=1+C \longrightarrow \boxed{C=1}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{3S+1}{(S-1)(S^2+1)} = \frac{2}{(S-1)} + \frac{-2S+1}{(S^2+1)}\right.$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{S-1} - \frac{2S}{S^2+1} + \frac{1}{S^2+1}\right\}$$

$$= 2e^t - 2\cos t + \sin t$$

« Long division method »

Note/ تستخدم طريقة القسمة الطويلة في حالة اذا كان أس في البسط اكبر من أو يساوي الاس في المقام.

Ex: find $\mathcal{L}^{-1}\left\{\dfrac{S^2+S+1}{S^2-5S+6}\right\}$

$$\begin{array}{r} 1 \\ S^2-5S+6 \overline{\big)\, \cancel{S^2}+S+1} \\ \underline{\cancel{S^2}-5S+6} \; - \\ 6S-5 \end{array}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{S^2+S+1}{S^2-5S+6}\right\} = \mathcal{L}^{-1}\left\{1+\frac{6S-5}{S^2-5S+6}\right\}$$

$$\frac{6S-5}{S^2-5S+6} = \frac{A}{(S-2)} + \frac{B}{(S-3)}$$

$$A = \left.\frac{6S-5}{S-3}\right]_{S=2} = \frac{12-5}{2-3} = \frac{7}{-1} = \boxed{-7}$$

$$B = \left.\frac{6S-5}{S-2}\right]_{S=3} = \frac{18-5}{3-2} = \frac{13}{1} = \boxed{13}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{S^2+S+1}{S^2-5S+6}\right\} = \mathcal{L}\left\{1+\frac{-7}{S-2}+\frac{13}{S-3}\right\}$$

$$= f(t) - 7e^{2t} + 13e^{3t}$$

Q/ $F(s) = \dfrac{5S^3+3S^2+8S+6}{S^2+4}$

{Hint: $(S^2+4) = (s+j2)(s-j2)$

Q/ $F(s) = \dfrac{5}{S^2(S+1)^3} = \dfrac{A}{S^2} + \dfrac{B}{S} + \dfrac{C}{(S+1)^3} + \dfrac{D}{(S+1)^2} + \dfrac{E}{(S+1)}$

$A = 5$, $B = -15$, $C = 5$

$D = 10$, $E = 15$