* Second Order D.E :

2nd order → homogenous

2nd order → non-homogenous

1- Homogenous D.E of 2nd order :-

$$\boxed{ay'' + by' + cy = 0} \Rightarrow$$ الصيغة العامة

خطوات الحل :-

١- ترتيب المعادلة المعطاة حسب الصيغة العامة ،

٢- نستخدم الرمز (r) بدلاً من كل مشتقة في المعادلة مثلاً $y'' = r^2$

$y' = r$

فتصبح المعادلة بالشكل الاتي $(ar^2 + br + c = 0)$

٣- نجد قيمة (r) أما بالتحليل (التجربة) أو عن طريق الدستور .

٤- بعد ايجاد (r) نطبق احد الشروط الثلاث :-

(A) IF $r_1 = r_2 \Rightarrow y = e^{rx}(c_1 + x c_2)$

(B) IF $r_1 \neq r_2 \Rightarrow y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

(C) IF r_1, r_2 is complex number $(a \mp bi)$

$$y = e^{\alpha x}(c_1 \cos Bx + c_2 \sin Bx$$

$r_{1,2} = \alpha \mp \beta i$

Examples: Solve the D.E

① $y'' + y' - 2y = 0$

$r^2 + r - 2 = 0$

$(r_1 + 2)(r_2 - 1) = 0$

$r_1 = -2 \quad , r_2 = 1$

$r_1 \neq r_2$

$\therefore \quad y = C_1 e^{-2x} + C_2 e^{x}$

② $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$(r - 2)(r - 2)$

$r_1 = r_2 = 2$

$\therefore \quad y = (C_1 + C_2 x) e^{2x}$

③ $y'' + 2y' + 5y = 0$

$r^2 + 2r + 5 = 0$

$r = \dfrac{-b \mp \sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow r = \dfrac{-2 \mp \sqrt{4 - 4 \times 1 \times 5}}{2}$

$r = -1 \mp 2i \quad , \quad r_{1,2} = \alpha \mp \beta i$

$\therefore \ y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

$y = e^{-x}[C_1 \cos(2x) + C_2 \sin(2x)]$

② Non-Homogenous D.E of 2nd order :-

$$ay'' + by' + cy = f(x) \Longrightarrow \text{الصيغة العامة}$$

* خطوات الحل :-

١- إيجاد y_h (عن طريق تطبيق الخطوات السابقة لـ Homo.)

٢- إيجاد y_p (وهناك عدة طرق) سيتم ذكرها.

٣- القانون : $\boxed{y = y_h + y_p}$

* طرق إيجاد (y_p) :-

١- اذا كان $f(x)$ ثابت فإن $y_p = A$ ثم نجد المشتقة (y''_p, y'_p) وتعوض في المعادلة المعطاة .

٢- اذا كان $f(x)$ كثيرة الحدود فتكون المعادلة لها حسب الأس ونشتق y_p لايجاد (y''_p, y'_p) وبعدها نعوض الـ A, B , ---- الخ ، في المعادلة ونجد y_p .

٣- اذا كان $f(x)$ عبارة عن (e^{ax}) فإن $y_p = A e^{ax}$ فنجد (y''_p, y'_p) .

٤- اذا كان $f(x)$ هي ($\cos ax$ or $\sin ax$) فان :-

$$y_p = A\cos ax + B\sin ax$$

ومن ثم ايجاد (y''_p, y'_p) ونعوض بالمعادلة ونجد A, B وبالتالي y_p .

٥- اذا كانت $f(x)$ هي ($\cosh ax$ or $\sinh ax$) فنقوم أولاً بالتحويل الاتي :-

$$\cosh ax = \frac{e^{ax} + e^{-ax}}{2} \quad , \quad \sinh ax = \frac{e^{ax} - e^{-ax}}{2}$$

وبعدها نطبق الخطوة رقم (٣) .

Ex: Solve:

① $y'' - 3y' - 10y = 3$

$y_h \Rightarrow$ $y'' - 3y' - 10y = 0$

$r^2 - 3r - 10 = 0$

$(r-5)(r+2) = 0$

$r_1 = 5 \quad , r_2 = -2 \quad , r_1 \neq r_2$

$$\boxed{y_h = c_1 e^{5x} + c_2 e^{-2x}}$$

$y_p \Rightarrow$ $y_p = A \rightarrow y'_p = 0 \rightarrow y''_p = 0 \rightarrow$ نعوض بالمعادلة بالسؤال

$0 - 3(0) - 10A = 3$

$$\boxed{A = \frac{-3}{10}}$$

$y = y_h + y_p$

$$\boxed{y = c_1 e^{+5x} + c_2 e^{-2x} - \frac{3}{10}}$$

Ex:

② $y'' + 3y' + 2y = x^2$

$y_h \Rightarrow$

$$y'' + 3y' + 2y = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$$r_1 = -1 \quad , r_2 = -2 \quad \Rightarrow r_1 \neq r_2$$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{-2x}}$$

$y_p \Rightarrow$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

→ تعوض بمعادلة السؤال

$$2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$$

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = x^2$$

$$2Ax^2 + (6A + 2B)x + 2A + 3B + 2C = x^2$$

* $2A\cancel{x^2} = \cancel{x^2}$

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

* $(6A + 2B)x = 0 \quad] \div x$

$$6A + 2B = 0$$

$$6A = -2B \quad \rightarrow \text{نعوض قيمة } A$$

$$6\left(\frac{1}{2}\right) = -2B$$

$$B = \frac{-3}{2}$$

* $2A + 3B + 2C = 0$

$$2\left(\frac{1}{2}\right) + 3\left(\frac{-3}{2}\right) + 2C = 0$$

$$1 - \frac{9}{2} + 2C = 0 \Rightarrow C = \frac{7}{4} \Rightarrow y_p = \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$$

$$y = y_h + y_p \Rightarrow y = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$$

3 Example

$$y'' + 6y' + 9y = 2e^{2x}$$

$y_h \Rightarrow$ $y'' + 6y' + 9y = 0$

$r^2 + 6r + 9 = 0$

$(r+3)(r+3) = 0 \Rightarrow r_1 = r_2 = -3$

$$y_h = e^{-3}(c_1 + xc_2)$$

$y_p \Rightarrow \begin{bmatrix} y_p = Ae^{2x} \\ y'_p = 2Ae^{2x} \\ y''_p = 4Ae^{2x} \end{bmatrix}$

$4Ae^{2x} + 6(2Ae^{2x}) + 9(Ae^{2x}) = 2e^{2x}$

$4Ae^{2x} + 12Ae^{2x} + 9Ae^{2x} = 2e^{2x}$

$25A\cancel{e^{2x}} = 2\cancel{e^{2x}}$

$$A = \frac{2}{25}$$

$\therefore y_p = Ae^{2x} \Rightarrow y_p = \frac{2}{25}e^{2x}$

$y = y_h + y_p$

$$y = e^{-3}(c_1 + xc_2) + \frac{2}{25}e^{2x}$$

Ex:
4 $y'' - y' = \cos(2x)$

$y_h \Rightarrow \quad y'' - y' = 0$

$r^2 - r = 0 \Rightarrow r(r-1) = 0 \quad , r_1 = 0 \; , r_2 = 1$

$y_h = C_1 e^0 + C_2 e^x$

$y_h = C_1 + C_2 e^x$

$y_p \Rightarrow [y_p = A\cos ax + B\sin ax] \rightarrow$ القانون

$y'_p = -Aa\sin ax + Ba\cos ax$

$y''_p = -Aa^2\cos ax - Ba^2\sin ax$

نعوض بالسؤال

$-Aa^2\cos ax - Ba^2\sin ax) - (-Aa\sin ax + Ba\cos ax) = \cos(2x)$

$-4A\cos(2x) - 4B\sin(2x) + 2A\sin(2x) - 2B\cos(2x) = \cos(2x)$ $\quad * a = 2$

$* \cos(2x)(-4A - 2B) = \cos(2x)$

$-4A - 2B = 1$ —— ①

$\sin(2x)(-4B + 2A) = 0$

$-4B + 2A = 0$ —— ② $\Rightarrow A = 2B \rightarrow$ sub in eq ①

$-4(2B) - 2B = 1$

$-8B - 2B = 1 \Rightarrow -10B = 1 \Rightarrow B = \frac{-1}{10} \; , A = \frac{-1}{5}$

$\therefore y_p = \frac{-1}{5}\cos ax - \frac{1}{10}\sin ax$

$y = y_h + y_p$

$y = C_1 + C_2 e^x - \frac{1}{5}\cos ax - \frac{1}{10}\sin ax$

5 $y'' - 9y = 4\cosh 2x$

$y_h \Rightarrow \quad y'' - 9y = 0$

$r^2 - 9 = 0 \Rightarrow (r+3)(r-3) = 0$

$r_1 \neq r_2 \quad , r_1 = -3 \quad , r_2 = 3$

$\boxed{y_h = c_1 e^{-3x} + c_2 e^{3x}}$

$f(x) = 4\cosh 2x = \cancel{4}^{2}\left(\frac{e^{2x} + e^{-2x}}{\cancel{2}}\right)$

$y_p \Rightarrow$

$= 2e^{2x} + 2e^{-2x}$

$\therefore \left[\begin{array}{l} y_p = Ae^{2x} + Be^{-2x} \\ y'_p = 2Ae^{2x} - 2Be^{-2x} \\ y''_p = 4Ae^{2x} + 4Be^{-2x} \end{array}\right]$

$4Ae^{2x} + 4Be^{-2x} - 9(Ae^{2x} + Be^{-2x}) = 2e^{2x} + 2e^{-2x}$

$4Ae^{2x} + 4Be^{-2x} - 9Ae^{2x} - 9Be^{-2x} = 2e^{2x} + 2e^{-2x}$

$* \cancel{e^{2x}}(4A - 9A) = 2\cancel{e^{2x}}$

$-5A = 2 \Longrightarrow A = \frac{-2}{5}$

$* \cancel{e^{-2x}}(4B - 9B) = 2\cancel{e^{-2x}}$

$-5B = 2 \Longrightarrow B = \frac{-2}{5}$

$\therefore y_p = -\frac{2}{5}e^{2x} - \frac{2}{5}e^{-2x}$

$y = y_h + y_p$

$y = c_1 e^{-3x} + c_2 e^{3x} - \frac{2}{5}\left(e^{2x} + e^{-2x}\right)$

Method of Undetermind Coefficients

f(x)	yp
x^n $(n=0,1,2)$	$Ax^n + Bx^{n-1} + \cdots + 0$
e^{ax}	Ae^{ax}
$\cos ax$ or $\sin ax$	$A\cos ax + B\sin ax$
$x^n \cdot e^{ax}$	$(Ax^n + \cdots + 0)e^{ax}$
$x^n \cos ax$, $x^n \sin ax$	$(Ax^n + \cdots + 0)\cos ax + (Bx^n + \cdots + 0)\sin ax$
$e^{ax}\cos ax$, $e^{ax}\sin ax$	$(Ae^{ax}\cos ax + Be^{ax}\sin ax)$
$x^n \cdot e^{ax} \cdot \cos ax$, $x^n e^{ax} \cdot \sin ax$ →	$(Ax^n + \cdots + 0)e^{ax}\cos ax + (Bx^n + \cdots + 0)e^{ax}\sin ax$

طرق ايجاد yp .

ملاحظة في حال المعادلة هي مجموع مربعين $(r^2 + a^2)$ فان الحل

سيكون $y_h = C_1 \sin(ax) + C_2 \cos(ax)$

Ex: $(r^2 + 9) \rightarrow y_h = C_1 \sin(3x) + C_2 \cos(3x)$

$(r^2 + 1) \rightarrow y_h = C_1 \sin x + C_2 \cos x$

Ex: Find y_p for each

ملاحظة ـ الرقم 16 الموجود امام الدالة ليس له اي تأثير على الحل ، لذلك يتم تجاهل اي ثوابت تضاعف الدالة بأكملها.

① $f(x) = 16e^{7x}\sin(10x)$

$\rightarrow e^{7x} = e^{7x}$

$\rightarrow \sin(10x) = A\cos(10x) + B\sin(10x)$

So $\rightarrow y_p = e^{7x}(A\cos(10x) + B\sin(10x))$

---

② $f(x) = (9x^2 - 103x)\cos x$

اهمل $9x^2 \rightarrow Ax^2 + Bx + C$

$\cos x \rightarrow D\cos x + E\sin x$

$\therefore y_p = (Ax^2+Bx+C)(\overset{A}{D}\cos x + \overset{B}{E}\sin x)$

$= \cos x(ADx^2 + BDx + CD) + \sin x(\underline{AE}x^2 + \underline{BE}x + \underline{CE})$

OR $\rightarrow y_p = (Ax^2+Bx+C)\cos x + (Dx^2+Ex+F)\sin x$

---

③ $f(x) = -e^{-2x}(3-5x)\cos(9x)$

$e^{-2x} \longrightarrow e^{-2x}$

$(3-5x) \longrightarrow Ax+B$

$\cos 9x \longrightarrow A\cos(9x) + B\sin(9x)$

$y_p = \underbrace{(Ax+B)e^{-2x}}_{A}\cos(9x) + \underbrace{(Cx+D)e^{-2x}}_{B}\sin(9x)$

① * تكملة المحاضرة السادسة :-

(2nd order D.E)

EX:

Solve: $4y'' + y = 5e^{x}\cos 2x$

SoL:- $r^2 + 1 = 0$

$\rightarrow y_h = c_1\cos(\frac{1}{2}x) + c_2\sin(\frac{1}{2}x)$

$\rightarrow y_p = e^{x}(A\cos 2x + B\sin 2x)$

$= Ae^{x}\cos 2x + Be^{x}\sin 2x$

قاعدة :- نقارن yp مع yh :-

- اذا كان هناك حد متشابة في (yh , yp) سوف نضرب معادلة yp * X

وهكذا الى ان لم يبقى حدود متشابهة.

Ex: $y'' + 4y' + 4y = 3xe^{-2x}$

$r^2 + 4r + 4 = 0 \Rightarrow (r+2)(r+2) = 0$

① $y_h = e^{-2x}(c_1 + xc_2)$

$y_h = c_1e^{-2x} + c_2xe^{-2x}$

② $y_p = (Ax + B)e^{-2x}$

$y_p = (Axe^{-2x} + Be^{-2x}) * x$ → * الان نقارن سوف نجد هناك حدود متشابهة

$y_p = (Ax^2e^{-2x} + Bxe^{-2x}) * x$

$y_p = Ax^3e^{-2x} + Bx^2e^{-2x}$ → * الان نتوقف حين لم يبقى حدود متشابهة

* في حالة هناك جمع او طرح دالتين ← نأخذ كل دالة على حدى ونجد y_p لها.

Ex: $y'' - 3y' + 2y = xe^{2x} + \sin x$

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0 \quad , r_1 = 2 , r_2 = 1$

$\boxed{y_h = c_1 e^{2x} + c_2 e^{x}}$

① $y'' - 3y' + 2y = xe^{2x}$

② $y'' - 3y' + 2y = \sin x$

$y_{p1} = (Ax+B)e^{2x}$

$= (Axe^{2x} + Be^{2x})x$ → نقارن

$= Ax^2e^{2x} + Bxe^{2x}$

$y_{p2} = C\cos x + D\sin x$

$\therefore y_p = y_{p1} + y_{p2}$

$= Ax^2e^{2x} + Bxe^{2x} + C\cos x + D\sin x$

Ex: $y'' + y = \sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$= \frac{1}{2} - \frac{1}{2}\cos 2x$

$y'' + y = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$(r^2 + 1 = 0) \Rightarrow \boxed{y_h = c_1\cos x + c_2 \sin x}$

① $y'' + y = \frac{1}{2}$

② $y'' + y = -\frac{1}{2}\cos(2x)$

$y_{p1} = A$

$y_{p2} = B\cos(2x) + C\sin(2x)$ → لا يوجد حدود متشابهة مع y_h

∴ لا يحتاج للتعديل

$y_p = y_{p1} + y_{p2}$

$= A + B\cos(2x) + C\sin(2x)$,

Ex: Solve:

$$y'' + 4y = x \sin x$$

Sol:

$r^2 + 4 = 0 \rightarrow y_h = C_1 \cos(2x) + C_2 \sin(2x)$

$y_p \Rightarrow$

$y_p = (A_1 x + A_0)\cos x + (B_1 x + B_0)\sin x$

$y'_p = -(A_1 x + A_0)\sin x + \cos x (A_1) + (B_1 x + B_0)(\cos x) + \sin x (B_1)$

نعوض بالسؤال

$y''_p = -(A_1 x + A_0)(\cos x) + (\sin x)(-A_1) - A_1 \sin x + (B_1 x + B_0)(-\sin x)$
$+ (\cos x)(B_1) + B_1 \cos x$

$y''_p = -(A_1 x + A_0)(\cos x) - 2A_1 \sin x + 2B_1 \cos x - (B_1 x + B_0)(\sin x)$

$-(A_1 x + A_0)(\cos x) - 2A_1 \sin x + 2B_1 \cos x - (B_1 x + B_0)(\sin x) + 4(A_1 x + A_0)\cos x$
$+ 4(B_1 x + B_0)\sin x$
$= x \sin x$

$3(A_1 x + A_0)\cos x + 3(B_1 x + B_0)\sin x - 2A_1 \sin x + 2B_1 \cos x = x \sin x$

فك الاقواس $\Rightarrow$

$\Rightarrow 3A_1 x \cos x + 3A_0 \cos x + 3B_1 x \sin x + 3B_0 \sin x - 2A_1 \sin x + 2B_1 \cos x = x \sin x$

$3B_1 \cancel{x \sin x} = \cancel{x \sin x}$

$3B_1 = 1 \Rightarrow \boxed{B_1 = \frac{1}{3}}$

$\cos x (3A_0 + 2B_1) = 0 \Rightarrow 3A_0 = -2B_1 \Rightarrow \boxed{A_0 = \frac{-2}{9}}$

$\sin x (3B_0 - 2A_1) = 0 \Rightarrow 3B_0 = 2A_1 \Rightarrow \boxed{B_0 = 0}$

$3A_1 x \cos x = 0 \Rightarrow \boxed{A_1 = 0}$

$\Rightarrow \boxed{y_p = \left(-\frac{2}{9} \cos x\right) + \frac{1}{3} x \sin x}$

$y = y_h + y_p$

$y = C_1 \cos(2x) + C_2 \sin(2x) - \frac{2}{9} \cos x + \frac{1}{3} x \sin x$

Ex: $y'' - 3y' + 2y = 14 \sin(2x) - 18 \cos(2x)$

$r^2 - 3r + 2 = 0 \rightarrow (r-2)(r-1) = 0$

$\boxed{y_h = C_1 e^{2x} + C_2 e^{x}}$

$y_p = A \cos(2x) + B \sin(2x)$

$y_p' = -2A \sin(2x) + 2B \cos(2x)$

$y_p'' = -4A \cos(2x) + (-4B \sin(2x))$

ملاحظة/ *

① في حالة الزوايا متشابهة نحل بمعادلة واحدة كما في هذا المثال.

② أما اذا كانت الزوايا الغير متشابهة فنأخذ معادلة yp لكل حد. فيصبح هنا $(y_{p_1} + y_{p_2})$.

نعوض بالسؤال :

$-4A\cos 2x - 4B \sin 2x - 3(-2A \sin 2x + 2B \cos 2x) + 2(A\cos 2x + B \sin 2x)$
$= 14 \sin(2x) - 18 \cos(2x)$

$\cos 2x(-4A - 6B + 2A) = -18 \cos 2x$

$-2A - 6B = -18 \Longrightarrow A = 9 - 3B$ —— ①

$\sin 2x(-4B + 6A + 2B) = 14 \sin 2x$

$6A - 2B = 14 \rightarrow$ نعوض قيمة A

$6(9 - 3B) - 2B = 14 \Longrightarrow 54 - 18B - 2B = 14 \rightarrow \boxed{B = 2}$

$A = 9 - 3(2) \Rightarrow \boxed{A = 3}$

$\boxed{y_p = 3 \cos(2x) + 2 \sin(2x)}$

$y = y_p + y_h$

$y = C_1 e^{2x} + C_2 e^{x} + 3 \cos(2x) + 2 \sin(2x)$

Ex: $y'' + 2y' = 2x + 5 - e^{-2x}$

$r^2 + 2r = 0$

$r(r+2) = 0 \rightarrow r_1 = 0 \ , \ r_2 = -2$

$y_h = c_1 e^0 + c_2 e^{-2x}$

$\boxed{y_h = c_1 + c_2 e^{-2x}}$

1 $\quad y'' + 2y' = 2x + 5$

2 $\quad y'' + 2y' = -e^{-2x}$

$y_{p1} = (Ax + B)x = Ax^2 + Bx$

$y_{p2} = (Ce^{-2x})x = Cxe^{-2x}$

$\boxed{y_p = Ax^2 + Bx + Cxe^{-2x}}$

$y'_p = 2Ax + B + Cx(-2e^{-2x}) + Ce^{-2x}$

$= 2Ax + B + Ce^{-2x} - 2Cxe^{-2x}$

$y''_p = 2A + (-2Ce^{-2x}) + (-2Cx)(-2e^{-2x}) + (e^{-2x})(-2C)$

$= 2A - 2Ce^{-2x} + 4Cxe^{-2x} - 2Ce^{-2x}$

$= 2A - 4Ce^{-2x} + 4Cxe^{-2x}$ } نعوض بالسؤال

$2A - 4Ce^{-2x} + \cancel{4Cxe^{-2x}} + 4Ax + 2B + 2Ce^{-2x} - \cancel{4Cxe^{-2x}} = 2x + 5 - e^{-2x}$

0 $\quad$ 0

$4Ax + 2A + 2B - 2Ce^{-2x} = 2x + 5 - e^{-2x}$

$4A\cancel{x} = 2\cancel{x} \Rightarrow \boxed{A = \frac{1}{2}}$

$2A + 2B = 5 \Rightarrow 2\left(\frac{1}{2}\right) + 2B = 5 \Rightarrow \boxed{B = \frac{5-1}{2}} \rightarrow \boxed{B = 2}$

$-2Ce^{-2x} = -e^{-2x} \Rightarrow 2C = 1 \Rightarrow \boxed{C = \frac{1}{2}}$

$y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$

EX: $y'' = 0 \quad , y(0) = 1 \quad , y(1) = 2$

$r^2 = 0 \Rightarrow r_1 = r_2 = 0$

$y_h = e^{0x}(C_1 + xC_2) = C_1 + xC_2$

$y_h = C_1 + xC_2$

$y(0) = 1 \Rightarrow 1 = C_1 + C_2(0) \Rightarrow \boxed{C_1 = 1}$

$y(1) = 2 \Rightarrow 2 = C_1 + C_2(1) \Rightarrow C_2 = 2 - C_2 \Rightarrow \boxed{C_2 = 1}$

$\therefore \boxed{y_h = 1 + x}$

Ex:

Solve:- $y'' + 2y' + 5y = e^{0.5x} + 40\cos 10x - 190\sin 10x$

$r^2 + 2r + 5 = 0 \Rightarrow r_{1,2} = -1 \mp 2i$

$$\boxed{y_h = e^{-x}(C_1 \cos 2x + C_2 \sin 2x)}$$

* For $e^{0.5x} \Rightarrow y_{p_1} = Ae^{0.5x}$

$$\left.\begin{array}{l} y_{p_1} = Ae^{0.5x} \\ y'_{p_1} = 0.5Ae^{0.5x} \\ y''_{p_1} = 0.25Ae^{0.5x} \end{array}\right\} \text{sub in question}$$

$0.25A + 2(0.5A) + 5A = 1$

$6.25A = 1 \Rightarrow A = \frac{1}{6.25} = \boxed{0.16}$ sub in y_{p_1}

$\therefore \boxed{y_{p_1} = 0.16\, e^{0.5x}}$

* For $(40\cos 10x - 190\sin 10x) \Rightarrow y_{p_2} = B\cos 10x + C\sin 10x$

$$\left.\begin{array}{l} y'_{p_2} = -10B\sin 10x + 10C\cos 10x \\ y''_{p_2} = -100B\cos 10x - 100C\sin 10x \end{array}\right\} \text{sub in quest}$$

$-100B\cos 10x - 100C\sin 10x - 20B\sin 10x + 20C\cos 10x + 5B\cos 10x + 5C\sin 10x$

$= 40\cos 10x - 190\sin 10x$

$\cancel{\cos 10x}(-100B + 20C + 5B) = 40\cancel{\cos 10x}$

$-95B + 20C = 40$ ——— ①

$\cancel{\sin 10x}(-100C - 20B + 5C) = -190\cancel{\sin 10x}$

$-95C - 20B = -190$

$B = \frac{190 - 95C}{20}$ ——— ② sub in ①

$-95\left(\frac{190 - 95C}{20}\right) + 20C = 40$

$-\frac{95 * 190}{20} + \frac{95 * 95C}{20} + 20C = 40$

$-902.5 + 471.25C = 40 \Rightarrow C = \frac{942.5}{471.25} = \boxed{2}$

$C = 2$

$$B = \frac{190 - (95 * 2)}{20} = \frac{190 - 190}{20} = 0$$

$\therefore \boxed{y_{p_2} = 2 \sin 10x}$

$\therefore y = y_h + y_{p_1} + y_{p_2}$

$$= e^{-x}(C_1 \cos 2x + C_2 \sin 2x) + 0.16 e^{0.5x} + 2 \sin 10x$$

Example /Write down the form of the particular solution to

$$y''+p(t)y'+q(t)y=g(t)$$

(a) $g(t) = 16\mathbf{e}^{7t}\sin(10t)$

Acos(10t)+Bsin(10t)

$$Y_P(t) = \mathbf{e}^{7t}\left(A\cos(10t) + B\sin(10t)\right)$$

(b) $g(t) = \left(9t^2 - 103t\right)\cos t$

$$Y_P(t) = \left(At^2 + Bt + C\right)\cos t + \left(Dt^2 + Et + F\right)\sin t$$

(c) $g(t) = -\mathbf{e}^{-2t}(3-5t)\cos(9t)$

This final part has all three parts to it. First, we will ignore the exponential and write down a guess for.

$$-(3-5t)\cos(9t)$$

The minus sign can also be ignored. The guess for this is

$$(At+B)\cos(9t) + (Ct+D)\sin(9t)$$

Now, tack an exponential back on and we're done.

$$Y_P(t) = \mathbf{e}^{-2t}(At+B)\cos(9t) + \mathbf{e}^{-2t}(Ct+D)\sin(9t)$$

Notice that we put the exponential on both terms.

Ex

$$y'' - 4y' - 12y = 3\mathbf{e}^{5t} + \sin(2t) + t\mathbf{e}^{4t}$$

$$Y_P(t) = -\frac{3}{7}\mathbf{e}^{5t} + \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t) - \frac{1}{36}(3t+1)\mathbf{e}^{4t}$$

Example 8 Write down the form of the particular solution to

$$y'' + p(t)y' + q(t)y = g(t)$$

for the following $g(t)$'s.

(a) $g(t) = 4\cos(6t) - 9\sin(6t)$

(b) $g(t) = -2\sin t + \sin(14t) - 5\cos(14t)$

(c) $g(t) = \mathbf{e}^{7t} + 6$

(d) $g(t) = 6t^2 - 7\sin(3t) + 9$

(e) $g(t) = 10\mathbf{e}^{t} - 5t\mathbf{e}^{-8t} + 2\mathbf{e}^{-8t}$

(f) $g(t) = t^2\cos t - 5t\sin t$

(g) $g(t) = 5\mathbf{e}^{-3t} + \mathbf{e}^{-3t}\cos(6t) - \sin(6t)$

Solution

a)

$$\underbrace{A\cos(6t)+B\sin(6t)}_{\text{guess for the cosine}}+\underbrace{C\cos(6t)+D\sin(6t)}_{\text{guess for the sine}}$$

To fix this notice that we can combine some terms as follows.

$$(A+C)\cos(6t)+(B+D)\sin(6t)$$

Upon doing this we can see that we've really got a single cosine with a coefficient and a single sine with a coefficient and so we may as well just use

$$Y_P(t)=A\cos(6t)+B\sin(6t)$$

b)

For this one we will get two sets of sines and cosines. This will arise because we have two different arguments in them. We will get one set for the sine with just a t as its argument and we'll get another set for the sine and cosine with the $14t$ as their arguments.

The guess for this function is

$$Y_P(t)=A\cos t+B\sin t+C\cos(14t)+D\sin(14t)$$

d)

This one can be a little tricky if you aren't paying attention. Let's first rewrite the function

$$\begin{aligned}g(t)&=6t^2-7\sin(3t)+9 \quad \text{as}\\ g(t)&=6t^2+9-7\sin(3t)\end{aligned}$$

All we did was move the 9. However, upon doing that we see that the function is really a sum of a quadratic polynomial and a sine. The guess for this is then

$$Y_P(t)=At^2+Bt+C+D\cos(3t)+E\sin(3t)$$

If we don't do this and treat the function as the sum of three terms we would get

$$At^2+Bt+C+D\cos(3t)+E\sin(3t)+G$$

e)

$$A\mathbf{e}^{t}+(Bt+C)\,\mathbf{e}^{-8t}+D\mathbf{e}^{-8t}$$

$$Y_P(t)=A\mathbf{e}^{t}+(Bt+C)\,\mathbf{e}^{-8t}$$

f)

$$Y_P(t)=\left(At^2+Bt+C\right)\cos t+\left(Dt^2+Et+F\right)\sin t$$

g)

$$Y_P(t)=A\mathbf{e}^{-3t}+\mathbf{e}^{-3t}\left(B\cos(6t)+C\sin(6t)\right)+D\cos(6t)+E\sin(6t)$$

H.W

(a) $y'' + 3y' - 28y = 7t + \mathbf{e}^{-7t} - 1$

(b) $y'' - 100y = 9t^2\mathbf{e}^{10t} + \cos t - t\sin t$

(c) $4y'' + y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

(d) $4y'' + 16y' + 17y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

(e) $y'' + 8y' + 16y = \mathbf{e}^{-4t} + \left(t^2 + 5\right)\mathbf{e}^{-4t}$