Exact Equation

* The general form is $M(x,y)+N(x,y)=0$
* There is one condition to said the equation is Exact

$$\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$ ← الشرط

اذا تحقق الشرط أعلاه، يجب اتباع الخطوات التالية:-

1. القانون العام لايجاد F: $\boxed{F=\int M(x,y)\,dx+g(y)}$

2. الاشتقاق الجزئي لـ F ($\frac{\partial f}{\partial y}$) بالنسبة لـ y فقط.

3. مساواة المعادلات $\frac{\partial f}{\partial y}=N(x,y)$ لايجاد $g'(y)$.

4. تكامل $g'(y)$ ← $\int g'(y)=g(y).dy$ لايجاد $g(y)$.

5. تعويض دالة $g(y)$ الناتجة من التكامل في الخطوة السابقة في المعادلة الناتجة من القانون العام في الخطوة الاولى (F).

Example : solve

$$\underbrace{\cos(x+y)}_{M}\,dx + (\underbrace{3y^2+2y+\cos(x+y)}_{N})\,dy = 0$$

$M(x,y) = \cos(x+y) \Rightarrow \dfrac{\partial M}{\partial y} = -\sin(x+y)$

$N(x,y) = 3y^2+2y+\cos(x,y) \Rightarrow \dfrac{\partial N}{\partial x} = -\sin(x+y)$

So $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow \therefore$ the equ. is Exact

① $F = \int M(x,y)\,dx + g(y)$ القانون العام

$F = \int \cos(x+y).dx + g(y)$

$F = \sin(x+y) + g(y)$ —①

② $\dfrac{\partial f}{\partial y} = \cos(x+y) + g'(y)$ مشتقة جزئية

③ $\dfrac{\partial f}{\partial y} = N(x,y)$ مساواة

$\cancel{\cos(x+y)} + g'(y) = 3y^2 + 2y + \cancel{\cos(x+y)}$

$g'(y) = 3y^2 + 2y$

④ $g(y) = \int g'(y)\,.dy$ تكامل

$g(y) = \int 3y^2 + 2y = y^3 + y^2$ → sub in eq ①

⑤ $F = \sin(x+y) + y^3 + y^2$ التعويض

$$\overset{M}{\frac{2x}{y^3}}\,dx + \overset{N}{\frac{y^2-3x^2}{y^4}}\,dy = 0$$

Ex: Solve:

$$M(x,y) = \frac{2x}{y^3} \Rightarrow \frac{\partial M}{\partial y} = \frac{-6x}{y^4}$$

$$N(x,y) = \frac{y^2-3x^2}{y^4} = \frac{y^2}{y^4} - \frac{3x^2}{y^4} \Rightarrow \frac{\partial N}{\partial x} = -\frac{6x}{y^4}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{the eq. is } \underline{\text{Exact}}$$

$$f = \int M(x,y)\,dx + g(y)$$

$$f = \int \frac{2x}{y^3}\,dx + g(y)$$

$$f = \frac{x^2}{y^3} + g(y) \quad \text{------} (1)$$

$$\frac{\partial f}{\partial y} = f_y = \frac{-3x^2}{y^4} + g'(y) \quad , f_y = \frac{\partial f}{\partial y} = N(x,y)$$

$$\cancel{\frac{-3x^2}{y^4}} + g'(y) = \frac{y^2}{y^4} - \cancel{\frac{3x^2}{y^4}}$$

$$\therefore \boxed{g'(y) = \frac{1}{y^2}}$$

$$g(y) = \int g'(y)\,dy = \int \frac{1}{y^2}\,dy = \frac{-1}{y} \quad \} \Rightarrow \text{sub in eq. (1) } \&$$

$$f = \frac{x^2}{y^3} - \frac{1}{y}$$

4 Linear Equation :-

the general Form is :

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$$

the Solution :

$$\boxed{y = \frac{\int Q(x)\, e^{\int p(x)dx} \cdot dx + C}{e^{\int p(x)dx}}}$$

Ex: Solve , $\frac{dy}{dx} + \frac{y}{x} = x^3$

$\frac{dy}{dx} + p(x)\,y = Q(x)$

$\therefore\ p(x) = \frac{1}{x}$, $Q(x) = x^3$

$\int p(x) = \int \frac{1}{x}\,dx = \ln x \Rightarrow e^{\int p(x)dx} = e^{\ln x} = x$

$$y = \frac{\int Q(x)\, e^{\int p(x)dx} \cdot dx + C}{e^{\int p(x)dx}}$$

$$y = \frac{\int (x^3 \cdot x)\cdot dx + C}{x}$$

$$y = \frac{\frac{x^5}{5} + C}{x} \Rightarrow y = \frac{x^5}{5x} + \frac{C}{x}$$

$$\boxed{y = \frac{x^4}{5} + \frac{C}{x}}$$

Ex: Solve $(1+x^2)dy + (2xy - 4x^2)dx = 0 \quad , y(0)=1$

$$(1+x^2)\,dy = -(2xy - 4x^2)dx$$

$$(1+x^2)\,dy = (4x^2 - 2xy)dx$$

$$\frac{dy}{dx} = \frac{4x^2 - 2xy}{1+x^2}$$

$$\frac{dy}{dx} = \frac{4x^2}{1+x^2} - \frac{2xy}{1+x^2}$$

the general form is $\left[\frac{dy}{dx} + P(x)y = Q(x)\right]$

$$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$$

$\therefore \boxed{P(x) = \frac{2x}{1+x^2}} \quad , \quad \boxed{Q(x) = \frac{4x^2}{1+x^2}}$

$$\int p(x)\,dx = \int \frac{2x}{1+x^2}\,dx \Rightarrow = \ln(1+x^2)$$

$$y = \frac{\int Q(x)\, e^{\ln(1+x)^2}\cdot dx + c}{e^{\ln(1+x^2)}}$$

$$y = \frac{\int \frac{4x^2}{\cancel{(1+x^2)}}\cdot\cancel{(1+x^2)}\,.dx + c}{1+x^2} \Rightarrow y = \frac{\int 4x^2\,.dx + c}{(1+x^2)}$$

$$y = \frac{\frac{4}{3}x^3 + c}{1+x^2} \quad , \text{ by Sub } x=0 , y=1$$

$$1 = \frac{0+c}{1+0} \Rightarrow c = 1$$

$$\therefore y = \frac{\frac{4}{3}x^3 + 1}{1+x^2} \Rightarrow \boxed{y = \frac{1}{1+x^2}\left(\frac{4}{3}x^3 + 1\right)}$$

Special cases of (linear D.E) :-

* $\frac{dy}{dx} + p(x).y = Q(x)$ ⟶ Linear equ.

* $\frac{dy}{dx} + p(x).y = Q(x).y^n$ ⟶ Bernoull's equ. $(n \neq 1)$

* $f(y).\frac{dy}{dx} + p(x).g(y) = Q(x)$ ⟶ non linear equ

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1 * Bernoull's equation

the generall form is → $\frac{dy}{dx} + p(x).y = Q(x)y^n$

خطوات الحل :

① ايجاد n

② فرض $z = y^{1-n}$

③ اشتقاق z $\left(\frac{dz}{dx}\right)$

④ ضرب المشتقة الناتجة بمعادلة السؤال لتنتج عنها معادلات جديدة تحتوي على المتغيرات (x,z) ويمكن حلها بطريقة (Linear) مباشرة .

⑤ استخدام قوانين Linear eq.

$$z = \frac{\int Q(x)\, e^{\int p(x)dx} . dx + c}{e^{\int p(x)dx}}$$

Ex : solve $\frac{dy}{dx} - y = xy^5$

solution

$n = 5$

$z = y^{1-n} \Rightarrow z = y^{1-5} = y^{-4}$

$\frac{dz}{dx} = -4y^{-5} \cdot \frac{dy}{dx}$

المشتقة الناتجة نضربها بمعادلة السؤال

$(-4y^{-5}) \cdot \frac{dy}{dx} - (-4y^{-5}) \cdot y = xy^5(-4y^{-5})$

$\frac{dz}{dx} + 4y^{-4} = -4x$ $\rightarrow$ Linear الآن يمكن حلها مباشرة

$$Z = \frac{\int Q(x) \cdot e^{\int p(x)dx} \cdot dx + C}{e^{\int p(x)dx}}$$

$p(x) = 4 \rightarrow \int p(x) \cdot dx = \int 4 \cdot dx = 4x$, $Q(x) = -4x$

$$Z = \frac{\int (-4x) \cdot e^{4x} \cdot dx + C}{e^{4x}}$$

$\int u dv = u \cdot v - \int v du$

$u = -4x \rightarrow du = -4\, dx$

$dv = e^{4x} \cdot dx \rightarrow v = \frac{1}{4} e^{4x}$

$= (-4x)(\frac{1}{4} e^{4x}) - \int (\frac{1}{4} e^{4x}) \cdot (-4dx)$

$= -x e^{4x} + e^{4x}$

نتيجة التكامل

$$\therefore \quad Z = \frac{-x e^{4x} + e^{4x} + c}{e^{4x}}$$

$$Z = \frac{-x e^{4x}}{e^{4x}} + \frac{e^{4x}}{e^{4x}} + \frac{c}{e^{4x}}$$

$$Z = -x + 1 + \frac{c}{e^{4x}}$$

$$\because \quad Z = y^{-4}$$

$$y^{-4} = -x + 1 + \frac{c}{e^{4x}} \quad \text{OR} \quad \frac{1}{y^4} = 1 - x + \frac{c}{e^{4x}}$$

Q/ Bernoulli's equa

$$y' + y = x y^2 \quad \longrightarrow \text{Ans: } \frac{1}{y} = x + 1 + c e^{x}$$

* Special case 2 (Linear D.E)

② A non-linear D.E of the form:

$$\boxed{f(y)\frac{dy}{dx}+P(x).g(y)=Q(x)}$$

* خطوات الحل :

١- كتابة الشكل العام للمعادلة لتحديد قيم المتغيرات .

٢- القانون / $\boxed{\frac{d}{dx}g(y)=f(y)\frac{dy}{dx}}$

٣- تعويض $V=g(y)$.

٤- ايجاد معادلة $\frac{dy}{dx}$ وتعويضها في معادلة السؤال .

٥- تكملة الحل كما في (Linear D.E) .

Ex: $\frac{1}{1+y^2}\frac{dy}{dx}+\frac{\tan^{-1}y}{x}=x$

$f(y)\frac{dy}{dx}+p(x).g(y)=Q(x) \Rightarrow$

$f(y)=\frac{1}{1+y^2}$

$p(x)=\frac{1}{x}$

$g(y)=\tan^{-1}y=V$

$Q(x)=x$

$\frac{d}{dx}g(y)=f(y)\frac{dy}{dx}$

$\frac{dv}{dx}=\frac{1}{1+y^2}\cdot\frac{dy}{dx}$

$\frac{dy}{dx}=(1+y^2)\frac{dv}{dx}$ ----Sub in question

$\frac{1}{\cancel{1+y^2}}\cdot\cancel{1+y^2}\frac{dv}{dx}+\frac{v}{x}=x$

$\therefore \frac{dv}{dx}+\frac{v}{x}=x$ ----- becam a linear D.E

$p(x)=\frac{1}{x}$, $Q(x)=x$

$\int p(x)dx=\int\frac{1}{x}dx=\ln x$

$$V = \frac{\int Q(x)\, e^{\int p(x)dx} \cdot dx + c}{e^{\int p(x)dx}}$$

$$V = \frac{\int x\, e^{\ln x}\, dx + c}{e^{\ln x}} \Rightarrow V = \frac{\int x^2\, dx + c}{x}$$

$$V = \frac{\frac{x^3}{3} + c}{x} = \frac{x^2}{3} + \frac{c}{x} = \tan^{-1} y$$

$$y = \tan\left[\frac{x^2}{3} + \frac{c}{x}\right]$$

H.W

1. $xy' + y + 4 = 0$
2. $x^3 - y\sin x + (\cos x + 2y)\frac{dy}{dx} = 0$
3. $y' = 3x^2(1+y^2)$
4. $2xy' = y - x$
5. $y' = e^{x-y}$
6. $(y^2-1)dx + (2xy - \sin y)dy = 0$

Questions

1) $xy - y' = y^4 e^{\frac{-3x^2}{2}}$

2) $3xy' + y + x^2y^4 = 0$

3) $x\,dy = (y^3 - 3y + 2)\,dx$

4) $y' + y = xy^2$

5) $\ln x \frac{dx}{dy} = \frac{x}{y}$

6) $x(2y-3)\,dx + (x^2+1)\,dy = 0$

7) $\sqrt{2xy}\quad y' = 1$

8) $x^2 dy + (y^2 - xy)\,dx = 0$

9) $xy' - y = \sqrt{x^2 - y^2}$

10) $(3x^2 + y\cos x)\,dx + (\sin x - 4y)\,dy$

11) $3xy' - y = \ln x + 1 \quad , y(1) = -2$

12) $x^3 - y\sin x + (\cos x + 2y)\frac{dy}{dx} = 0$

13) $y' = e^{x-y}$

14) $xy' + y + 4 = 0$

Q/ $(1-x^2)\,y' + xy = 2x$

$$, y' = \frac{dy}{dx}$$

$$(1-x^2)\frac{dy}{dx} + xy = 2x \quad] \div (1-x^2)$$

$$\frac{dy}{dx} + \frac{x}{(1-x)^2}\cdot y = \frac{2x}{(1-x^3)}$$

$$p(x) = \frac{x}{(1-x^2)} \quad , \quad Q(x) = \frac{2x}{(1-x^2)} \quad \Longrightarrow \text{Linear eq.}$$

$$\int p(x)\,dx = \int \frac{x}{(1-x^2)}\cdot dx = \frac{-1}{2}\ln(1-x^2) = \ln(1-x^2)^{-\frac{1}{2}}$$

$$y = \frac{\int \frac{2x}{(1-x^2)}\cdot e^{\ln(1-x^2)^{-1/2}}\cdot dx + c}{e^{\ln(1-x^2)^{-1/2}}}$$

$$y = \frac{\int \frac{2x}{(1-x^2)(1-x^2)^{1/2}}\cdot dx + c}{(1-x^2)^{-1/2}}$$

نكامل البسط على حدى أولا ثم نعيده الى المعادلة :—

$$\int \frac{2x}{(1-x^2)(1-x^2)^{1/2}}\cdot dx = \int 2x\,(1-x^2)^{-\frac{3}{2}}\cdot dx \quad —①$$

Let:- $u = 1-x^2$, $du = -2x\,dx \Rightarrow \boxed{dx = \frac{du}{-2x}}$ sub in the eq ①

$$\int \cancel{2x}(1-x)^{-3/2}\cdot\frac{du}{-\cancel{2x}} = -\int (1-x)^{-3/2}\,du$$

Now we can sub (u) value

$$-\int u^{-3/2}\cdot du = -\frac{u^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C = 2u^{-1/2} = 2(1-x^2)^{-1/2}$$

So:-

$$y = \frac{2(1-x^2)^{-1/2} + C}{(1-x^2)^{-1/2}} = \boxed{2 + C\sqrt{1-x^2}}$$

أسئلة هامة

Q/ $\underbrace{\left[x\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)\right]}_{M(x,y)}dx+\underbrace{\left[x\cos\left(\frac{y}{x}\right)\right]}_{N(x,y)}dy=0$

$$\frac{\partial M}{\partial y}=\cancel{\cos\frac{y}{x}-\cos\frac{y}{x}}+\frac{y}{x}\sin\left(\frac{y}{x}\right)=\frac{y}{x}\sin\left(\frac{y}{x}\right)$$

$$\frac{\partial N}{\partial x}=\cos\frac{y}{x}-x\sin\frac{y}{x}\cdot\left(\frac{-y}{x^2}\right)=\cos\frac{y}{x}+\frac{y}{x}\sin\frac{y}{x}$$

(NOT Exact)!!

$$\frac{dy}{dx}=\frac{y\cos\left(\frac{y}{x}\right)-x\sin\left(\frac{y}{x}\right)}{x\cos\frac{y}{x}}=f(x,y) \quad \text{homo.}$$

divide by x

$$\frac{dy}{dx}=\frac{\frac{y}{x}\cos\left(\frac{y}{x}\right)-\frac{x}{x}\sin\left(\frac{y}{x}\right)}{\frac{x}{x}\cos\left(\frac{y}{x}\right)}=\frac{\frac{y}{x}\cos\left(\frac{y}{x}\right)-\sin\left(\frac{y}{x}\right)}{\cos\frac{y}{x}}$$

$$v=\frac{y}{x}\Rightarrow\frac{dy}{dx}=\frac{v\cos v-\sin v}{\cos v}$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\therefore\ v+x\frac{dv}{dx}=\frac{v\cos v-\sin v}{\cos v}$$

$$x\frac{dv}{dx}=\cancel{v}-\frac{\sin v}{\cos v}\cancel{-v}\Rightarrow x\frac{dv}{dx}=\frac{\sin v}{\cos v}$$

$$\int\frac{\cos v}{\sin v}dv=\int\frac{dx}{x}+c$$

$$[\ln\sin v=\ln x+c \quad \} \text{expoment}$$

$$\sin v=x+e^{c}$$

$$\sin\frac{y}{x}=x+e^{c}\Rightarrow\boxed{y=x\sin^{-1}(x+e^{c})}$$

(توضيح اشتقاق السؤال السابق)

$$\frac{\partial M}{\partial y} = x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)$$

$$= \left[\cancel{x}\cos\left(\frac{y}{x}\right)\cdot\frac{1}{\cancel{x}}\right] - \left[y\cdot\left(-\frac{1}{x}\sin\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\cdot 1\right]$$

$$= \cancel{\cos\left(\frac{y}{x}\right)} + \frac{y}{x}\sin\left(\frac{y}{x}\right) - \cancel{\cos\left(\frac{y}{x}\right)} = \frac{y}{x}\sin\left(\frac{y}{x}\right)$$

$$\frac{\partial N}{\partial x} = x\cos\left(\frac{y}{x}\right)$$

$$= \left[\cancel{x}\cdot\left(-\sin\left(\frac{y}{x}\right)\right)\left(\frac{-y}{x^{2}}\right)\right] + \left[\cos\left(\frac{y}{x}\right)\cdot(1)\right]$$

$$= \frac{y}{x}\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)$$

linear

Q/ $\left[3x\frac{dy}{dx} - y = \ln x + 1\right] \div 3x$ $y(1) = -2$

$$\frac{dy}{dx} - \frac{y}{3x} = \frac{\ln x + 1}{3x}$$

$$p(x) = \frac{-1}{3x} \quad , \quad Q(x) = \frac{\ln x + 1}{3x}$$

$$\int p(x)\,dx = \int \frac{-1}{3x}\,dx = \frac{-1}{3}\ln x = \ln x^{\frac{-1}{3}}$$

$$y = \frac{\int \frac{\ln x+1}{3x} \cdot x^{-\frac{1}{3}}\,dx + c}{x^{-\frac{1}{3}}} \Rightarrow$$

تكامل البسط على حدة

$$\therefore \int \frac{\ln x+1}{3x} \cdot \frac{1}{x^{\frac{1}{3}}}\,dx \Rightarrow \therefore \int \frac{\ln x+1}{3x^{\frac{4}{3}}}\,dx$$

$\int u\,dv = uv - \int v\,du \Rightarrow$ integration by parts

$$u = \ln x + 1 \quad , \quad dv = 3x^{-\frac{4}{3}}$$

$$du = \frac{1}{x}\,dx \quad , \quad v = -9x^{-\frac{1}{3}}$$

$$1 - \frac{4}{3} = \frac{-1}{3}$$

$$-\frac{1}{3} - 1 =$$

$$\therefore (\ln x+1)(-9x^{-\frac{1}{3}}) + \int 9x^{-\frac{1}{3}} \cdot \frac{1}{x}\,dx \quad] \div -9$$

$$= (\ln x+1)(x^{-\frac{1}{3}}) - \int x^{-\frac{4}{3}}\,dx$$

$$= (\ln x+1)(x^{-\frac{1}{3}}) + 3x^{-\frac{1}{3}} = x^{-\frac{1}{3}}(\ln x + 1 + 3)$$

Sub in y

$$y = \frac{x^{-\frac{1}{3}}(\ln x+4) + c}{x^{-\frac{1}{3}}} = (\ln x + 4) + x^{\frac{1}{3}}c$$

at $x = 1$, $y = -2$

$$-2 = (\underbrace{\ln 1}_{0}) + 4 + c \Rightarrow c = -6$$

$$\boxed{y = \ln x + 4 - 6x^{\frac{1}{3}}}$$