« Partial Derivatives »

الاشتقاق الجزئي

when the function f has more the one varialbe like F(x,y,z), can derivated partialy.

$F(x,y)$:

- $F_x = \dfrac{\partial f}{\partial x}$ (المشتقة الاولى بالنسبة لـ x)
- $F_y = \dfrac{\partial f}{\partial y}$ (المشتقة الاولى بالنسبة لـ y)
- $F_{xx} = \dfrac{\partial^2 f}{\partial x^2}$. (مشتقة اولى لـ x وثم مشتقة ثانية لـ x أيضاً)
- $F_{yy} = \dfrac{\partial^2 f}{\partial y^2}$ (مشتقة اولى لـ y ، وثم مشتقة ثانية لـ y وأيضاً)
- $F_{xy} = \dfrac{\partial^2 f}{\partial y\, \partial x}$ (مشتقة اولى لـ x ، وثم مشتقة اخرى لـ y للدالة Fx)
- $F_{yx} = \dfrac{\partial^2 f}{\partial x\, \partial y}$ (مشتقة اولى لـ x ، وثم مشتقة اخرى لـ x للدالة Fy)
- $F_{yyx} = \dfrac{\partial^3 f}{\partial x\, \partial y^2}$
- $F_{yyxx} = \dfrac{\partial^4 f}{\partial x^2\, \partial y^2}$

Ex: $F = 2x^3y^4 + 5x^2y^3 + 7x^2 + 15y^2$, find f_x, f_y

f_{xx}, f_{yy}

f_{xy}, f_{yx}

Sol.: ① $f_x = \frac{\partial f}{\partial x} = 6y^4x^2 + 10y^3x + 14x$

② $F_y = \frac{\partial f}{\partial y} = 8x^3y^3 + 15x^2y^2 + 30y$

③ $F_{xx} = 12y^4x + 10y^3 + 14$

④ $F_{yy} = 24x^3y^2 + 30x^2y + 30$

⑤ $F_{yx} = 24y^3x^2 + 30y^2x$

⑥ $F_{xy} = 24x^2y^3 + 30xy^2$

Ex: Find F_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$

Sol:

$f_y = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4$

* Chain Rule :-

$w = f(x,y,z)$

① If x, y, z are a functions of "t", then:- $x(t), y(t), z(t)$:-

$$\boxed{\frac{\partial F}{\partial t} = F_x \cdot \frac{\partial x}{\partial t} + F_y \cdot \frac{\partial y}{\partial t} + F_z \cdot \frac{\partial z}{\partial t}}$$

② If x, y, z are a functions of $\underline{r}, \underline{s}$, that :-

$x = g(r,s)$, $y = h(r,s)$, $z = k(r,s)$

$$\frac{\partial F}{\partial r} = F_r = F_x \cdot \frac{\partial x}{\partial r} + F_y \cdot \frac{\partial y}{\partial r} + F_z \cdot \frac{\partial z}{\partial r}$$

$$\frac{\partial F}{\partial s} = F_s = F_x \cdot \frac{\partial x}{\partial s} + F_y \cdot \frac{\partial y}{\partial s} + F_z \cdot \frac{\partial z}{\partial s}$$

Ex: Using chain rule, Find $\frac{\partial w}{\partial t}$ For $(w = xy)$, $x = \cos t$, $y = \sin t$ what is the derivative value at $t = \pi/2$.?

$w = xy$

x and y are a function of "t" then

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial w}{\partial x} = y$$

$$\frac{\partial x}{\partial t} = -\sin t$$

$$\frac{\partial w}{\partial y} = x \quad , \quad \frac{\partial y}{\partial t} = \cos t$$

→

then :- $\frac{\partial w}{\partial t} = y(-\sin t) + x(\cos t)$

$$\frac{\partial w}{\partial t} = x\cos t - y\sin t$$

* الان نعوض الدوال x and y في المعادلة ، المشتقة اعلاه :-

$$\frac{\partial w}{\partial t} = \cos t . \cos t - \sin t . \sin t$$

$$\frac{\partial w}{\partial t} = \cos^2 t - \sin^2 t$$

$$\frac{\partial w}{\partial t} = \cos 2t$$

* at $t = \frac{\pi}{2} \rightsquigarrow \frac{\partial w}{\partial t} = \cos 2\left(\frac{\pi}{2}\right) = \boxed{-1}$

Ex: Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if

$w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$

$x, y, z \rightarrow$ Function of (r, s)

$$\therefore \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$\frac{\partial w}{\partial x} = 1$, $\frac{\partial x}{\partial r} = \frac{1}{s}$

$\frac{\partial w}{\partial y} = 2$, $\frac{\partial y}{\partial r} = 2r$

$\frac{\partial w}{\partial z} = 2z$, $\frac{\partial z}{\partial r} = 2$

$$\therefore \frac{\partial w}{\partial r} = (1)\cdot\frac{1}{s} + (2)\cdot 2r + (2z)\cdot 2$$

$$= \frac{1}{s} + 4r + 4z$$

$$= \frac{1}{s} + 4r + 4(2r)$$

$$\boxed{\frac{\partial w}{\partial r} = \frac{1}{s} + 12r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$\frac{\partial x}{\partial s} = -\frac{r}{s^2}$$

$$\frac{\partial y}{\partial s} = \frac{1}{s}$$

$$\frac{\partial z}{\partial s} = 0$$

$$\therefore \frac{\partial w}{\partial s} = (1)\cdot\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0)$$

$$\boxed{\frac{\partial w}{\partial s} = \frac{-r}{s^2} + \frac{2}{s}}$$

* If $w=f(x)$ and $x=g(r,s)$ then

$$\boxed{\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}\\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s}\end{aligned}}$$

* Questions:

① If $F(x,y)=e^{xy}$, $x=r\cos\theta$, $y=r\sin\theta$ } Find: F_r , F_θ

② $F(x,y)=x^2-xy^2$, $x=\cos(t)^2$, $y=\ln(t^2)$ } Find F_t

③ $F(x,y,z)=x^2y^2+yz^3+\sin(xz)$, $x=\ln t^2$, $y=e^{t^2}$, $z=(\tan t^2)^2$

Find F_t .

* Implicit Differentiation :-

الاشتقاق الضمني

A Formula of implicit differentiation is :

$$\boxed{\frac{dy}{dx} = \frac{-F_x}{F_y}}$$ → For 2-variable

Suppose that $F(x,y)$ is differentiable and that equation $F(x,y)=0$ defines y as a differentiable function of x. then at any point where $F_y \neq 0$.

Ex: Use theorm of implicit differentiation to find dy/dx For the function of $y^2 - x^2 - \sin xy = 0$

Sol/

$$F(x,y) = y^2 - x^2 - \sin xy$$

$$F_x = -2x - y\cos xy$$

$$F_y = 2y - x\cos xy$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

$$\frac{dy}{dx} = \frac{-(-2x - y\cos xy)}{2y - x\cos xy}$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

* For more than 2-variable

$$\boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$

Ex: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0,0,0)$ if $x^3 + z^2 + ye^{xz} + z\cos y = 0$

Sol: $F(x,y,z) = x^3 + z^2 + ye^{xz} + z\cos y$

$$F_x = 3x^2 + yze^{xz}$$

$$F_y = e^{xz} - z\sin y$$

$$F_z = 2z + xye^{xz} + \cos y.$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-3x^2 - yze^{xz}}{2z + xye^{xz} + \cos y}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-e^{xz} + z\sin y}{2z + xye^{xz} + \cos y}$$

at $p(0,0,0)$ we get:

$$\frac{\partial z}{\partial x} = \boxed{0} \quad , \quad \frac{\partial z}{\partial y} = \boxed{-1}$$

* Directional drivative and gradient :-

∇f : gradient vector

$D_u f$: directioal derivative

$$\boxed{\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k}$$

$$\boxed{D_u f\Big|_{P_0} = \vec{\nabla f}\Big|_{P_0} \cdot \vec{u}}$$

<u>Theorm</u>: The directional derivative is a dot product.

$\vec{u}$: is a unit vector

<u>Ex</u>: Find directional drivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\vec{V} = 3i - 4j$.

sol/: $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

$\frac{\partial f}{\partial x} = F_x = e^y - y\sin(xy) \Longrightarrow F_{x P_0} = e^0 - 0 \cdot \sin(0) = \boxed{1}$

$\frac{\partial f}{\partial y} = F_y = xe^y - x\sin(xy) \Longrightarrow F_{y P_0} = 2 \cdot e^0 - 2\sin(0) = \boxed{2}$

$$\boxed{\nabla f = i + 2j}$$

$\vec{u} = \frac{\vec{V}}{|V|} = \frac{3i - 4j}{\sqrt{(3)^2 + (4)^2}} = \frac{3}{5} i - \frac{4}{5} j$

$\therefore D_u f\Big|_{P_0} = \vec{\nabla f}\Big|_{P_0} \cdot \vec{u} = (i + 2j) \cdot \left(\frac{3}{5} i - \frac{4}{5} j\right) = \frac{3}{5} - \frac{8}{5} = \boxed{-1}$

* Using angles in Directional derivatives

If the direction of directional derivative is described by giving the angle θ of inclination of the vector u, then we can use the expression $\left(\vec{u} = \cos\theta\, i + \sin\theta\, j\right)$

$$\therefore D_u f = \vec{\nabla} f \cdot [\cos\theta\, i + \sin\theta\, j]$$

Ex: Find directional derivative of $f(x,y) = e^{xy}$ at the point $P_o(-2,0)$ at vector make angle of $\left(\frac{\pi}{3}\right)$ with positive x-axis.

Sol: $\vec{u} = \cos\theta\, i + \sin\theta\, j$

$$= \cos\left(\frac{\pi}{3}\right) i + \sin\left(\frac{\pi}{3}\right) j \qquad \frac{\pi}{3} = 60^\circ$$

$$\boxed{\vec{u} = \frac{1}{2} i + \frac{\sqrt{3}}{2} j} \text{ unit vector}$$

$$F_x = \frac{\partial f}{\partial x} = y e^{xy} \Longrightarrow F_{x_{P_o}} = 0 \cdot e^{-2} = \boxed{0}$$

$$F_y = \frac{\partial f}{\partial y} = x e^{xy} \longrightarrow F_{y_{P_o}} = -2 \cdot e^{0} = \boxed{-2}$$

$$\therefore \vec{\nabla f} = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\boxed{\vec{\nabla} f = -2j}$$

$$\therefore D_u f_{(-2,0)} = \vec{\nabla} f_{(-2,0)} \cdot \vec{u}$$

$$= (-2j) \cdot \left(\frac{1}{2} i + \frac{\sqrt{3}}{2} j\right) = \boxed{-\sqrt{3}}$$

* Extreme values and Saddle Points.

① if $D > 0$ $\begin{cases} f_{xx} > 0 \rightarrow \text{minimum} \\ f_{xx} < 0 \rightarrow \text{maximum} \end{cases}$

② if $D < 0 \longrightarrow$ saddle point

③ if $D = 0 \longrightarrow$ test failed.

$$\boxed{D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}}$$, $f_{xy} = f_{yx}$

Ex① Find the local extreme valuse of the function :-

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

Sol:

$f_x = y - 2x - 2 = 0$ ——— ①

$f_y = x - 2y - 2 = 0$ - - - ②

$\rightarrow y = 2 + 2x \longrightarrow$ sub in eq ②

Solving equations

$x - 2(2+2x) - 2 = 0$

$x - 4 - 4x - 2 = 0$

$-3x = 2 + 4$

$\boxed{x = -2}$

$y = 2 + 2(-2)$

$\boxed{y = -2}$

the point $(-2,-2)$ is critical Point.

$F_{xx} = -2$

$f_{yy} = -2$

$f_{xy} = 1 = f_{yx}$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$D = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4-1 = \boxed{3 > 0}$$

$\therefore D > 0$ and $f_{xx} < 0 \Rightarrow$

$\therefore$ the $F(x,y)$ has alocal maximum Point at $(-2,-2)$

EX: ② $f(x,y) = x^3 + y^3 - 3xy + 15$, Find local extreme Points.

Sol: $F_x = 3x^2 - 3y = 0$ ---① $\}\rightarrow y = x^2$. sub in ②

$F_y = 3y^2 - 3x = 0$ ---②

$3(x^2)^2 - 3x = 0$

$3x^4 - 3x = 0 \Rightarrow 3x(x^3 - 1) = 0 \rightarrow x_1 = 0,\ x_2 = 1$

$y = x^2 \rightarrow y_1 = x_1^2 = 0,\ y_2 = x_2^2 = 1$

$\therefore$ There are two critical Points $(0,0)$, $(1,1)$

$F_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$, $f_{yx} = -3$

$$D_{(0,0)} = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = 0 - 9 = -9 < 0$$

$\therefore D < 0 \rightarrow$ Saddle points.

$$D_{(1,1)} = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 36 - 9 = 27 > 0$$

$\therefore D > 0$, $f_{xx} > 0 \Rightarrow$ min. point

* Tangent planes :-

$$\boxed{F_{x_{P_0}} \cdot (x - x_0) + F_{y_{P_0}} \cdot (y - y_0) + F_{z_{P_0}} \cdot (z - z_0) = 0}$$

Normal Line:

- $x = x_0 + F_{x_{P_0}} \cdot t$
- $y = y_0 + F_{y_{P_0}} \cdot t$
- $z = z_0 + F_{z_{P_0}} \cdot t$

Ex: Find the tang. plane & normal line. of the surfase of $x^2 + y^2 + z - 9 = 0$ at point $P_0(\underset{x_0,\, y_0,\, z_0}{1, 2, 4})$.

Sol: $F_x = 2x \longrightarrow F_{x_{P_0}} = 2$

$F_y = 2y \longrightarrow F_{y_{P_0}} = 4$

$F_z = 1 \longrightarrow F_{z_{P_0}} = 1$

*. the tangent is:

$$F_{x_{P_0}}(x - x_0) + F_{y_{P_0}}(y - y_0) + F_{z_{P_0}}(z - z_0) = 0$$

$$2(x-1) + 4(y-2) + (z-4) = 0$$

$$2x - 2 + 4y - 8 + z - 4 = 0$$

$$\boxed{2x + 4y + z = 14}$$

*. normal line is:-

$$\boxed{\begin{aligned} x &= 1 + 2t \\ y &= 2 + 4t \\ z &= 4 + t \end{aligned}}$$

theorm of tangent Plane

* IF $Z = F(x,y)$

$$F_{x_{p_0}}(x - x_0) + F_{y_{p_0}}(y - y_0) - (z - z_0) = 0$$

Ex: Find tang. plane of $Z = x\cos y - ye^{x}$ at P_0 (0,0)

Sol:

$F_x = \cos y - ye^{x} \Rightarrow F_{x_{p_0}} = \cos(0) - 0 \cdot e^{0}$

$= 1 - 0 = \boxed{1}$

$F_y = -x\sin y - e^{x} \Rightarrow F_{y_{p_0}} = -0 \cdot \sin(0) - e^{0}$

$= 0 - 1 = \boxed{-1}$

∴ tangent plane is

$1(x - 0) + (-1)(y - 0) - (z - 0) = 0$

$$\boxed{x - y - z = 0}$$