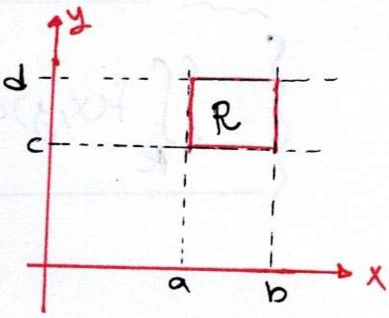
Lect. 4

"Multiple Integral"

* Double integral

let (x,y) be a continuous Intervel of the variable on a rectangular region (R) in x-y plane.



* Let $A(x) = \int_c^d f(x,y)\,dy$ and $B(y) = \int_a^b f(x,y)\,dx$

it is clear that $A(x)$ = is continuous function of x on $[a,b]$

and " " $B(y)$ " " " " y on $[c,d]$

- So, we can find $\int_a^b A(x)\,dx$ and $\int_c^d B(y)\,dy$

$$\therefore \int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \iint_R f(x,y)\,dA$$

By the same way:

$$\int_c^d B(y)\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$$

where $dA = dx\,dy = dy\,dx$

theorm:

if $f(x,y)$ is continuous throughout the rectangular region:

$$R: \rightarrow \quad a \leq x \leq b$$
$$c \leq y \leq d$$

then:

$$\boxed{\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy}$$

Ex: Find $I = \int_0^1 \int_0^2 (x^2 + xy)\,dy\,dx$

$$I = \int_0^1 \left[\int_0^2 (x^2+xy)\,dy \right] dx$$

$$= \int_0^1 \left[x^2 y + \frac{x}{2} y^2 \right]_0^2 \cdot dx$$

$$= \int_0^1 [2x^2 + 2x] \cdot dx = \left[\frac{2}{3} x^3 + x^2 \right]_0^1 = \frac{2}{3} + 1 = \boxed{\frac{5}{3}}$$

* if we calculate reverse integral

$$I_1 = \int_0^2 \int_0^1 (x^2 + xy)\,\cancel{dy}\,dx \cdot \cancel{dx}\,dy \rightarrow \int_0^2 \left[\frac{x^3}{3} + \frac{x^2}{2} y \right]_0^1 \cdot dy$$

$$= \int_0^2 \left[\frac{1}{3} + \frac{1}{2} y \right] dy$$

$$= \left[\frac{1}{3} y + \frac{1}{2*2} y^2 \right]_0^2 = \frac{2}{3} + \frac{(2)^2}{4} = \frac{2}{3} + 1 = \boxed{\frac{5}{3}}$$

$\therefore$ same solution

Lecture -4-

Multiple Integrals

- Double integrals:

Example: Calculate

$$\iint_R f(x,y)dA$$

for $f(x,y)=1-6x^2y$ and R: $0\le x\le 2$, $-1\le y\le 1$

Solution: by fubini's theorem:

$$\iint_R f(x,y)dA=\int_{-1}^{1}\int_{0}^{2}(1-6x^2y)dxdy=\int_{-1}^{1}\left[x-\frac{6x^3y}{3}\right]_{x=0}^{x=2}dy$$

$$=\int_{-1}^{1}[x-2x^3y]_{x=0}^{x=2}dy=\int_{-1}^{1}(2-(2)(2)^3y)dy$$

$$=\int_{-1}^{1}[2-16y]dy$$

$$=\left[2y-\frac{16y^3}{2}\right]_{-1}^{1}=[2y-8y^2]_{-1}^{1}$$

$$=\left((2)(1)-(8)(1)^2\right)-\left((2)(-1)-(8)(-1)^2\right)$$

$$=(2-8)-(-2-8)$$

$$=(-6)-(-10)$$

$$=-6+10=4$$

Reversing the order of integration gives the same answer

$$= \int_0^2 \int_{-1}^1 (1 - 6x^2y)dydx = \int_0^2 \left[y - \frac{6x^2y^2}{2}\right]_{y=-1}^{y=1} dx = \int_0^2 [y - 3x^2y^2]_{y=-1}^{y=1} dx$$

$$= \int_0^2 [(1 - 3x^2) - (-1 - 3x^2)]\, dx$$

$$= \int_0^2 (1 - 3x^2 + 1 + 3x^2)\, dx$$

$$= \int_0^2 2\, dx = [2x]_0^2$$

$= (2)(2) - 0 = 4$

Example: Evaluate the integral

$$\int_0^3 \int_1^2 (1 + 8xy)dydx$$

Solution:

$$= \int_0^3 \int_1^2 (1 + 8xy)dydx = \int_0^3 \left[y + \frac{8xy^2}{2}\right]_{y=1}^{y=2} dx$$

$$= \int_0^3 [y + 4xy^2]_{y=1}^{y=2} dx$$

$$= \int_0^3 [(2 + (4)(x)(2)^2) - (1 + (4)(x)(1)^2)]\, dx$$

$$= \int_0^3 [(2 + 16x) - (1 + 4x)]\, dx = \int_0^3 (2 + 16x - 1 - 4x)dx$$

$$= \int_0^3 (1 + 12x)dx = \left[x + \frac{12x^2}{2}\right]_0^3$$

$$= [x + 6x^2]_0^3 = (3 + (6)(3)^2 - 0)$$

$$= 3 + 54 = 57$$

Example: Evaluate the integral

$$\int_0^1 \int_0^2 (x + 3)dydx$$

Solution:

$$\int_0^1 \int_0^2 (x + 3)dydx = \int_0^1 [xy + 3y]_{y=0}^{y=2} dx$$

$$= \int_0^1 (2x + (3)(2) - 0)dx$$

$$= \int_0^1 (2x + 6)dx$$

$$= \left[\frac{2x^2}{2} + 6x\right]_0^1 = [x^2 + 6x]_0^1$$

$$= [1 + (6)(1) - 0]$$

$$= 1 + 6 = 7$$

: Evaluate the integral

1. $\int_{0}^{3}\int_{-2}^{0}(x^2y-2xy)dydx$

2. $\int_{-1}^{0}\int_{-1}^{1}(x+y+1)dxdy$

3. $\int_{1}^{3}\int_{-1}^{1}(2x-4y)dydx$

4. $\int_{2}^{4}\int_{0}^{1}x^2ydxdy$

5. $\int_{\pi}^{2\pi}\int_{0}^{\pi}(sinx+cosy)dxdy$

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* double integral ove bounded, Non Rectangular region:

① if $\underline{R}$:
$$a \leq x \leq b$$
$$g_1(x) \leq y \leq g_2(x)$$

$$\therefore \iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \cdot dy \cdot dx$$ قانون

② if $\underline{R}$:
$$c \leq y \leq d$$
$$h_1(y) \leq x \leq h_2(y)$$

$$\therefore \iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \cdot dy$$ قانون

Ex:

Find $I = \int_0^1 \int_0^x xy \, dy \, dx$

$$I = \int_0^1 \left[\frac{x}{2} y^2\right]_0^x \, dx$$

$$I_1 = \int_0^1 \left[\frac{x^3}{2}\right] dx$$

$$= \frac{x^4}{8}\Big]_0^1 = \frac{1}{8}$$

Ex: $I = \iint_R (x^2+3y)\, dA$.

when: ① R in xy-plane bounded by the curve $y = x^2$ and $y = x$

② R in xy-plane bounded by $y = x^2$ and $y = x+2$

Sol/

* الخطوة الاولى / رسم المعادلات على المحور الاحداثي

* الخطوة الثانية / نأخذ شريحة عمودية أو أفقية (سيكون الحل نفسه)

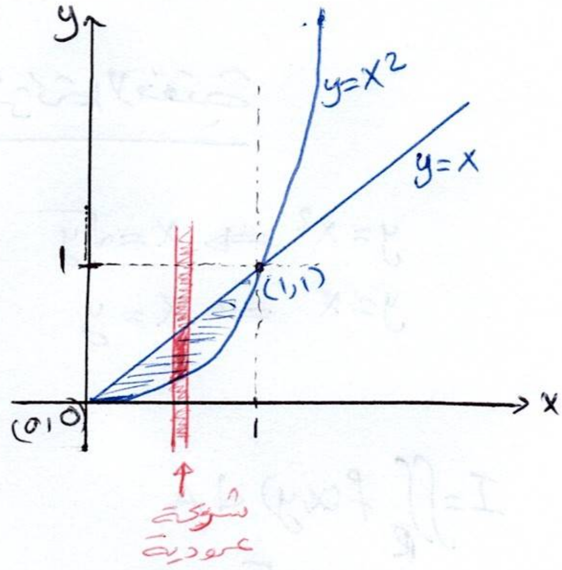
① ملاحظة / عند أخذ شريحة عمودية ستكون حركة الشريحة على امتداد المحور (x)، وبذلك تكون حدود التكامل بالنسبة لـ (x) ثابتة [a,b] . اما حدود (y) متغيرة [معادلة] .

② ملاحظة / عند اخذ شريحة أفقية ستكون حركة الشريحة على امتداد المحور (y) ، وبذلك تكون حدود التكامل بالنسبة لـ (y) ثابتة [c,d] ، اما حدود (x) متغيرة .

① SOL //

طريقة الشريحة عمودية

$$y = x^2 \quad , y = x$$



$$I = \iint_R (x^2+3y)\, dA$$

$$= \int_0^1 \int_{y=x^2}^{y=x} (x^2+3y)\, dy \cdot dx$$

$$= \int_0^1 \left[x^2 y + \frac{3}{2} y^2 \right]_{x^2}^{x} \cdot dx$$

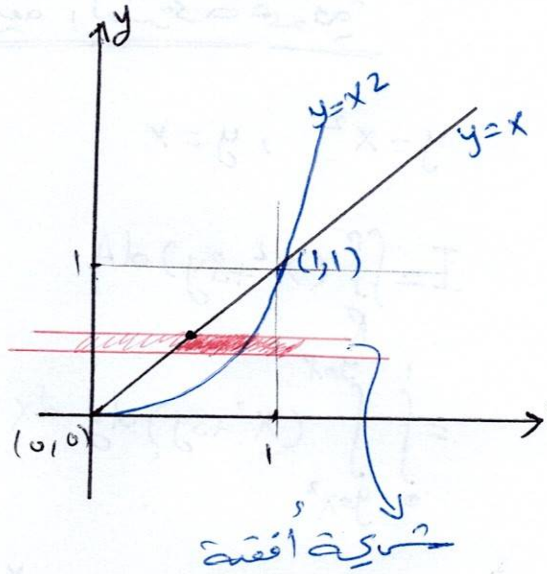
$$= \int_0^1 \left[\left(x^3 + \frac{3}{2}x^2\right) - \left(x^4 + \frac{3}{2}x^4\right) \right] \cdot dx$$

$$= \int_0^1 \left(x^3 + \frac{3}{2}x^2 - x^4 - \frac{3}{2}x^4\right) dx$$

$$= \int_0^1 \left(x^3 + \frac{3}{2}x^2 - \frac{5}{2}x^4\right) \cdot dx = \left[\frac{x^4}{4} + \frac{x^3}{2} - \frac{x^5}{2} \right]_0^1$$

$$= \frac{1}{4} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} = \boxed{\frac{1}{4}}$$

طريقة الشريحة الافقية



$$y = x^2 \Rightarrow x = \sqrt{y}$$
$$y = x \Rightarrow x = y$$

$$I = \iint_R f(x,y)\, dA$$

$$I = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} (x^2 + 3y)\, dx \cdot dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + 3yx \right]_y^{\sqrt{y}} \cdot dy$$

$$= \int_0^1 \left[\frac{((y)^{\frac{1}{2}})^3}{3} + 3y(y^{\frac{1}{2}}) \right] - \left[\frac{y^3}{3} + 3y^2 \right] \cdot dy$$

$$= \int_0^1 \left(\frac{y^{3/2}}{3} + 3y^{3/2} - \frac{y^3}{3} - 3y^2 \right] \cdot dy$$

$$= \int_0^1 \left(\frac{10}{3} y^{3/2} - \frac{y^3}{3} - 3y^2 \right) \cdot dy = \left[\frac{\cancel{10}^{2} * 2}{3 * \cancel{5}_{1}} y^{5/2} - \frac{y^4}{12} - y^3 \right]_0^1$$

$$= \frac{4}{3} - \frac{1}{12} - \frac{1}{1} = \frac{16 - 1 - 12}{12} = \frac{3}{12} = \boxed{\frac{1}{4}}$$

* نفس الناتج السابق ---

② $R: \rightarrow y = x^2 \ , \ y = x + 2$

باستخدام شريحة عمودية

$$I = \iint_R f(x,y)\, dA$$

$$= \int_{-1}^{2} \int_{y=x^2}^{y=x+2} (x^2 + 3y)\, dy \cdot dx$$

$$= \int_{-1}^{2} \left[x^2 y + \frac{3}{2} y^2 \right]_{x^2}^{x+2} \cdot dx$$

$$= \int_{-1}^{2} \left[x^2(x+2) + \frac{3}{2}(x+2)^2 \right] - \left[x^4 + \frac{3x^4}{2} \right] dx$$

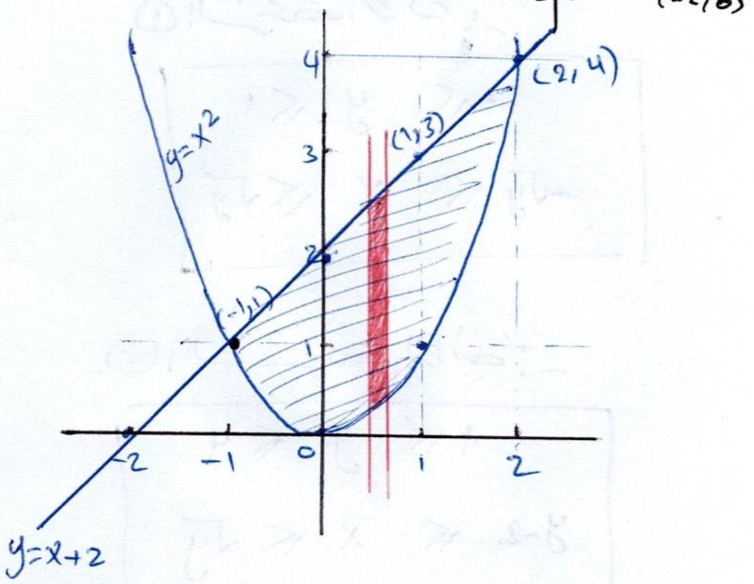
= and continue — —

$y = x^2$

x	y	
0	0	(0,0)
1	1	(1,1)
2	4	(2,4)

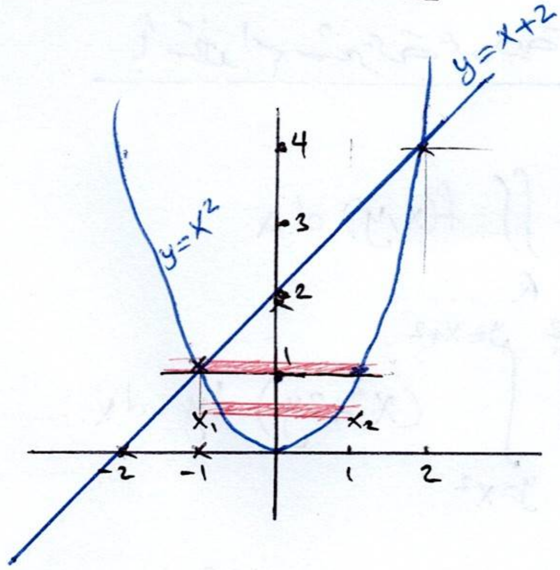
$y = x + 2$

x	y	
0	2	(0,2)
1	3	(1,3)
2	4	(2,4)
-1	1	(-1,1)
-2	0	(-2,0)



H.w أكمل الحل

طريقة الشريحة الافقية



* في هذه الحالة تقسم المساحة الى قسمين ، لذلك نحتاج الى شريحتين افقيتين .

① الشريحة الاولى :-

$$\boxed{\begin{array}{c} 0 \le y \le 1 \\ -\sqrt{y} \le x \le \sqrt{y} \end{array}}$$

② الشريحة الثانية :-

$$\boxed{\begin{array}{c} 1 \le y \le 4 \\ y-2 \le x \le \sqrt{y} \end{array}}$$

$$\therefore \quad I = \int_{y=0}^{y=1} \int_{-\sqrt{y}}^{+\sqrt{y}} (x^2+3y)\, dx\, dy + \int_{1}^{4} \int_{x=y-2}^{x=\sqrt{y}} (x^2+3y)\, dx\, dy$$

Ex: Evaluate $\iint_R x^2\, dA$, where R is the Region bounded by: $xy=16$, $x=8$, $y=0$, $y=x$?

Sol: → الشريحة العمودية

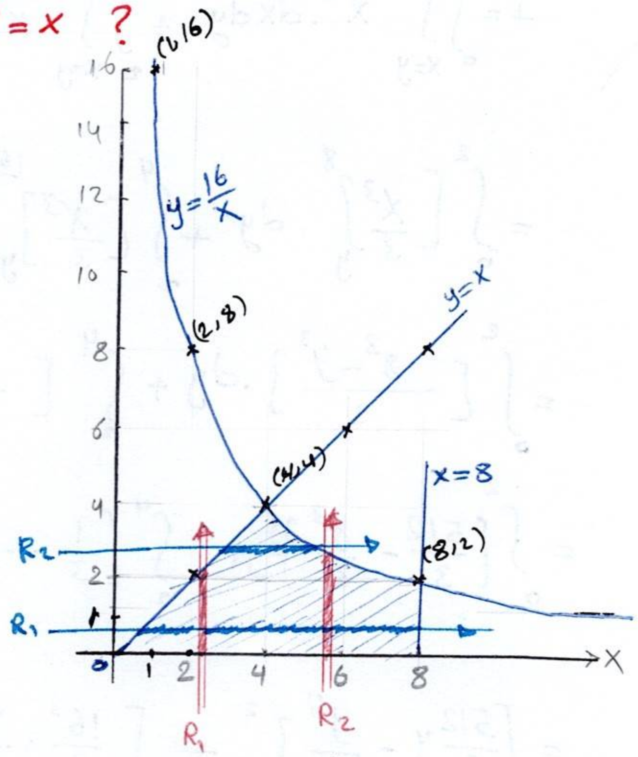
$xy=16$

$y=\frac{16}{x}$

x	y	
0	0	(0,0)
1	16	(1,16)
2	8	(2,8
4	4	(4,4)
8	2	(8,2)
16	1	(16,1)

$y=x$

x	y	
0	0	(0,0)
1	1	(1,1)
2	2	(2,2)
3	3	(3,3)
4	4	(4,4)
5	5	(5,5)



$$*\; I=\iint_{R_1} x^2\, dA+\iint_{R_2} x^2\, dA$$

$$=\int_0^4\int_{y=0}^{y=x} x^2 .dy.dx+\int_4^8\int_0^{\frac{16}{x}} x^2 .dy.dx$$

$$=\int_0^4 [x^2y]_0^x .dx+\int_4^8 [x^2y]_0^{\frac{16}{x}} .dx$$

$$=\int_0^4 x^3 .dx+\int_4^8 \cancel{x^2}.\frac{16}{\cancel{x}} .dx$$

$$=\left[\frac{x^4}{4}\right]_0^4+\left[\frac{\cancel{16}\,x^2}{\cancel{2}}\right]_4^8$$

$$=\left(\frac{(4)^4}{4}\right)+\left[\frac{8(8^2)}{1}-\frac{8(4^2)}{1}\right]$$

$$=\frac{256}{4}+512-128 \quad =\frac{1792}{4}=\boxed{448}$$

*

الشريحة الافقية

$$I = \int_0^2 \int_{x=y}^{x=8} x^2 . dx dy + \int_2^4 \int_{x=y}^{x=\frac{16}{y}} x^2 . dx . dy$$

$$= \int_0^2 \left[\frac{x^3}{3} \right]_y^8 . dy + \int_2^4 \left[\frac{x^3}{3} \right]_y^{\frac{16}{y}} . dy$$

$$= \int_0^2 \left[\frac{8^3 - y^3}{3} \right] . dy + \int_2^4 \left[\frac{(\frac{16}{y})^3}{3} - \frac{y^3}{3} \right] . dy$$

$$= \int_0^2 \left[\frac{512}{3} - \frac{y^3}{3} \right] dy + \int_2^4 \left[\frac{1}{3} . \frac{16^3}{y^3} - \frac{y^3}{3} \right] . dy$$

$$= \left[\frac{512}{3} y - \frac{y^4}{4 \times 3} \right]_0^2 + \left[\frac{16^3}{3} . \frac{y^{-2}}{-2} - \frac{y^4}{4 \times 3} \right]_2^4$$

$$= \left[\frac{1024}{3} - \frac{\cancel{16}^{\,4}}{\cancel{12}_{\,3}} \right] + \left[\frac{-4096}{6 \, y^2} - \frac{y^4}{12} \right]_2^4$$

$$= \left[\frac{1020}{3} \right] + \left[\left(\frac{-4096}{6(4)^4} - \frac{(4^4)}{12} \right) - \left(\frac{-4096}{6(2)^2} - \frac{(2^4)}{12} \right) \right]$$

$$= 340 + \left[\frac{-4096}{96} - \frac{256}{12} + \frac{4096}{24} + \frac{16}{12} \right]$$

$$= 340 + 108 = \boxed{448}$$

Ex: Reverse the order and evaluate the integral

$$I = \int_{1}^{4} \int_{\frac{x}{4}}^{\sqrt{x}} xy \, dy \cdot dx \qquad ?$$

Sol:

ملاحظة/ المطلوب في السؤال هو عكس اتجاه الشريحة
وذلك بتغيير الشريحة من عمودية الى
أفقية ، بعد الرسم ، سوف تتغير حدود
التكامل بحسب اتجاه الشريحة.

* يتضح من السؤال أن الشريحة المستخدمة هي شريحة عمودية $(dy\,dx)$
يجب تغييرها الى شريحة أفقية (dx, dy).

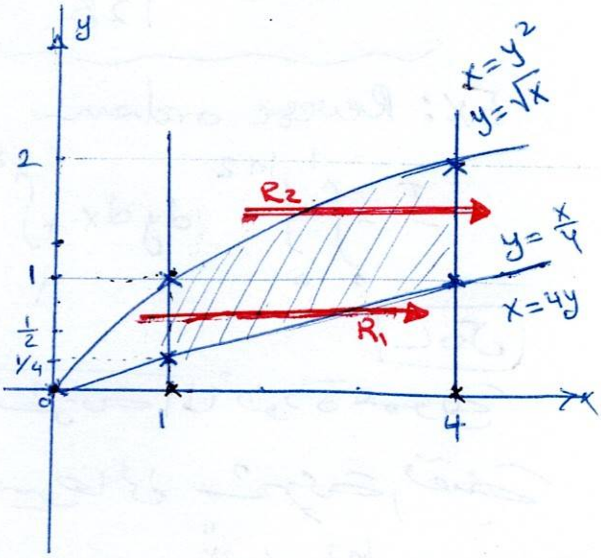
$1 \leq x \leq 4$

$y = \sqrt{x}$

x	y
0	0
1	1
4	2

$y = \frac{x}{4}$

x	y
0	0
1	$\frac{1}{4}$
4	1



نستنتج الرسم من حدود التكامل
المعطاة في السؤال

نحتاج الى شريحتين أفقيتين في الحل ،

$$I = \iint_{R_1} xy \cdot dx\,dy + \iint_{R_2} xy \cdot dx\,dy$$

$$\therefore I = \int_{\frac{1}{4}}^{1} \int_{1}^{4y} xy \cdot dx\,dy + \int_{1}^{2} \int_{x=y^2}^{x=4} xy \cdot dx\,dy$$

$$= \int_{\frac{1}{4}}^{1} \left[x^2 y \right]_{1}^{4y} \cdot dy + \int_{1}^{2} \left[x^2 y \right]_{y^2}^{4} \cdot dy$$

$$= \int_{\frac{1}{4}}^{1} \left[8y^3 - \frac{y}{2}\right] \cdot dy + \int_{1}^{2} \left[8y - \frac{y^5}{2}\right] \cdot dy$$

$$= \left[2y^4 - \frac{y^2}{4}\right]_{\frac{1}{4}}^{1} + \left[4y^2 - \frac{y^6}{12}\right]_{1}^{2}$$

$$= \left[2 - \frac{1}{4} - \frac{2}{(16)^2} + \frac{1}{4 \times 16}\right] + \left[16 - \frac{64}{12} - 4 + \frac{1}{12}\right]$$

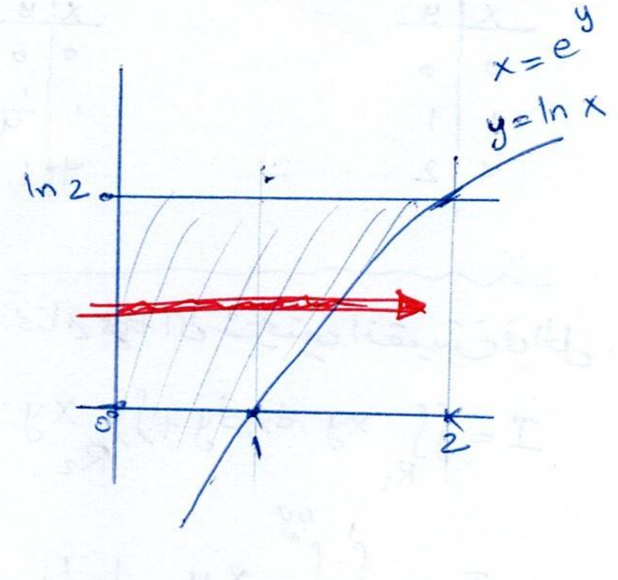
$$= \left[\frac{7}{4} + \frac{1}{128}\right] + \left[\frac{12}{1} - \frac{\cancelto{21}{63}}{\cancelto{4}{12}}\right] = \frac{224 + 1 + 1536 - 672}{128}$$

$$\underline{\underline{32}}$$

$$= \frac{1089}{128}$$

Ex: Revese orden

$$I = \int_{0}^{1}\int_{0}^{\ln 2} dy\, dx + \int_{1}^{2}\int_{\ln x}^{\ln 2} dy\, dx$$



$\boxed{\text{SoL}}$

يصبح الشريحة المأخوذة عمودية

يجب تغييرها الى شريحة افقية

$$I = \int_{0}^{\ln 2}\int_{x=0}^{x=e^y} dx.dy \qquad = \int_{0}^{\ln 2} [x]_{0}^{e^y} \cdot dy$$

$$= \int_{0}^{\ln 2} e^y . dy \Rightarrow = \left[e^y\right]_{0}^{\ln 2} \qquad = \cancel{e}^{\cancel{\ln 2}} - e^0$$

$$= 2 - 1 = \boxed{1}$$

* Triple Integral

* triple integral in rectangular region

$$\boxed{\iiint_B f(x,y,z)\,dv = \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx}$$

Ex: Evaluate $\iiint xyz^2 . dv$, where B is the rectangular box
$B: \{0 \leqslant x \leqslant 1 \ , \ -1 \leqslant y \leqslant 2 \ , \ 0 \leqslant z \leqslant 3\}$

Sol:
$$\iiint_B xyz^2 . dv = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 . dx\,dy\,dz$$

$$= \int_0^3 \int_{-1}^2 \left[\frac{x^2}{2} yz^2\right]_0^1 . dy\,dz$$

$$= \int_0^3 \int_{-1}^2 \frac{1}{2} yz^2 . dy\,dz$$

$$= \int_0^3 \left[\frac{y^2}{4} z^2\right]_{-1}^2 . dz = \int_0^3 \left[\frac{4}{4} z^2 - \frac{(-1)^2}{4} z^2\right] . dz$$

$$= \int_0^3 \left(z^2 - \frac{1}{4} z^2\right) dz = \int_0^3 \frac{3}{4} z^2 . dz$$

$$= \left[\frac{\cancel{3}}{4} \cdot \frac{z^3}{\cancel{3}}\right]_0^3 = \frac{(3)^3}{4} = \frac{27}{4}$$