((Lect. ~~7~~))

((Fourier Series))

* هو طريقة تتيح كتابة اي دالة رياضية دورية في شكل متسلسلة أو مجموع من الدوال من دوال sin و cos مضروب بمعامل معين.

* طريقة ايجاد period :-

① بحالة وجود فترات

$$\boxed{\text{Period} = P_{\text{أكبر}} - P_{\text{أصغر}}}$$

ex①: $f(x) = x^2 \quad , \quad 0 < x < \pi$

$$P = \pi - 0 = \boxed{\pi}$$

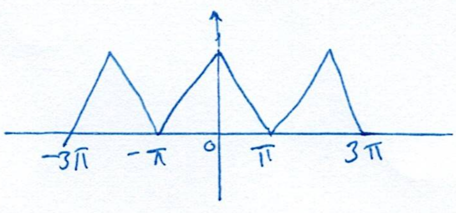
ex②: $f(x) = \begin{cases} 0 & , -5 < x < 0 \\ 3 & , 0 < x < 5 \end{cases}$

$$P = 5 - (-5) = \boxed{10}$$

ex③: $f(x) = \left.\begin{cases} -x & , -1 < x < 0 \\ x & 0 < x < 1 \\ 1 & 1 < x < 3 \end{cases}\right\} \quad P = 3 - (-1) = \boxed{4}$

② بحالة وجود رسم :-

ex



$$P = \pi - (-\pi) = 2\pi$$

ملاحظات لتسهيل الحل /

① $\sin n\pi = 0 \quad \rightarrow \sin(\pi) = \sin(2\pi) = \sin(3\pi) = 0$

② $\cos n\pi = (-1)^n \begin{cases} +1 & \text{even} \\ -1 & \text{odd} \end{cases}$

③ $\sin 2n\pi = 0$

④ $\cos 2n\pi = 1$

⑤ $\sin((2n \pm 1)\pi) = 0$

⑥ $\cos((2n \mp 1)\pi) = -1$

⑦ $\sin\left(\frac{(2n-1)}{2}\pi\right) = (-1)^{n+1}$

⑧ $\cos\left(\frac{(2n-1)}{2}\pi\right) = 0$

⑨ $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \rightarrow n: \text{even} \\ +1 & \rightarrow n = 1, 5, 9 \\ -1 & \rightarrow n = 3, 7, 11 \end{cases}$ زوجي

⑩ $\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \rightarrow n: \text{odd} \\ +1 & \rightarrow n = 4, 8, 12 \\ -1 & \rightarrow n = 2, 6, 10 \end{cases}$ فردي

⑪ $\cos(-x) = \cos x$

⑫ $\sin(-x) = -\sin x$

* معاملات رئيسية للفورير (Euler formula)

القانون العام

$$f(x) = A + \sum_{n=1}^{\infty} [a_n \cos(n w_o x) + b_n \sin(n w_o x)]$$

* $w_o = 2\pi f$, $f = \frac{1}{P}$ $\quad$ p : الفترة periodic = T

$= 2\pi \frac{1}{P}$

$w_o = \frac{2\pi}{P}$

* $A = \frac{1}{2} a_0 = \frac{1}{2} \frac{1}{\text{half period}} \int_{\text{الفترة الاصغر}}^{\text{الفترة الاكبر}} f(x)$ → قانون A

* $a_n = \frac{1}{\text{half Period}} \int_{\text{فترة الاصغر}}^{\text{فترة الاكبر}} f(x) . \cos n w_o x . dx$ → قانون a_n

* $b_n = \frac{1}{\text{half period}} \int_{\text{فترة الاصغر}}^{\text{فترة الاكبر}} f(x) . \sin n w_o x . dx$ → قانون b_n

half period → نصف الفترة

$P = 2\pi$ → half period $= \pi$

Ex: Find fourier Series of

$$f(x)=\begin{cases}1 & 0\leqslant x\leqslant \pi \\ 0 & -\pi\leqslant x\leqslant 0\end{cases}$$

Sol

$$P=\pi-(-\pi)=\pi+\pi=2\pi$$

$$w_0=\frac{2\pi}{P}=\frac{2\pi}{2\pi}=1$$

$$* \quad A=\frac{1}{2}a_0=\frac{1}{2}\cdot\frac{1}{half\,P.}\int_{-\pi}^{\pi}f(x)\,.\,dx$$

(n=0) تعويض

$$A=\frac{1}{2}\frac{1}{\pi}\left[\int_0^{\pi}(1)+\cancel{\int_{-\pi}^{0}(0)}\right].dx$$

$$A=\frac{1}{2\pi}\int_0^{\pi}(1)\,.\,dx$$

$$A=\frac{1}{2\pi}[x]_0^{\pi} \Rightarrow A=\frac{1}{2\pi}[\pi-0]=\boxed{\frac{1}{2}}$$

$$a_n=\frac{1}{half.P.}\int_{\text{اصغر}}^{\text{اكبر}}f(x)\cos nx\,dx$$

$$a_n=\frac{1}{\pi}\left[\int_0^{\pi}(1)\cos nx\,dx+\cancel{\int_{-\pi}^{0}(0)\cos nx\,dx}\right]$$

$$a_n=\frac{1}{\pi}\int_0^{\pi}\cos nx\,dx$$

$$=\frac{1}{\pi}\cdot\frac{1}{n}[\sin nx]_0^{\pi}$$

$$=\frac{1}{\pi n}[\cancel{\sin n\pi}-\cancel{\sin n(0)}] \Rightarrow \boxed{a_n=0}$$

$$* \; b_n = \frac{1}{\text{half } P.} \int_{\text{اصغر}}^{\text{أكبر}} f(x) \sin(nx) \cdot dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$b_n = \frac{1}{\pi} \left[\int_{0}^{\pi} (1) \sin nx \, dx + \cancel{\int_{-\pi}^{0} (0) \sin nx \, dx}^{\,0} \right]$$

$$b_n = \frac{1}{\pi} \cdot \frac{1}{n} \left[-\cos nx \right]_0^{\pi}$$

$$b_n = \frac{1}{\pi n} \left[-\cos(\pi n) + \cos(0) \right]$$

$$b_n = \frac{1}{n\pi} \left[-(-1)^n + 1 \right]$$

فردي $n=1 \Longrightarrow b_n = \frac{1}{n\pi}\left[-(-1)^1 + 1\right] = \frac{2}{n\pi}$

زوجي $n=2 \Longrightarrow b_n = \frac{1}{2\pi}\left[-(-1)^2 + 1\right] = 0$

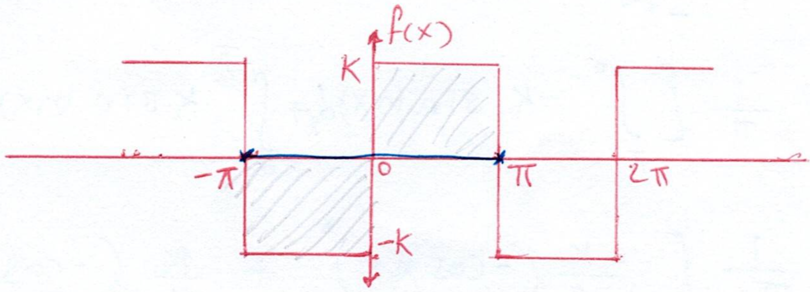
المعادلة الرئيسة

$$\therefore \; f(x) = A + \cancel{\sum_{n=1}^{\infty} a_n \cos n\omega_0 x}^{\,0} + \sum_{n=1}^{\infty} b_n \sin n\omega_0 x$$

$$f(x) = \frac{1}{2} + \sum_{\text{odd}}^{\infty} \frac{2}{n\pi} \sin nx$$

$$f(x) = \frac{1}{2} + \left(\frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \cdots \right)$$

Example: Find fourier coefficients of the periodic function $f(x)$.



Solution:

$$f(x) = A + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$p = \pi - (-\pi) = 2\pi$$

$$\omega = \frac{2\pi}{p} = \frac{2\pi}{2\pi} = 1$$

$$A = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot dx$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} (-k)\, dx + \int_{0}^{\pi} (k) \cdot dx \right]$$

$$= \frac{1}{2\pi} \left([-kx]_{-\pi}^{0} + [kx]_{0}^{\pi} \right)$$

$$= \frac{1}{2\pi} \left(-k(0+\pi) + k(\pi - 0) \right]$$

$$= \frac{1}{2\pi} [-k\pi + k\pi] = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \cdot dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} (-k) \cos(nx)\, dx + \int_{0}^{\pi} (k) \cos(nx) \cdot dx$$

$$= \frac{1}{\pi} \left[\frac{-k}{n} (\sin nx)_{-\pi}^{0} + \frac{k}{n} (\sin nx)_{0}^{\pi} \right]$$

$$= \frac{1}{\pi} [0 + 0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin(nx).dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -K\ \sin(nx)dx + \int_{0}^{\pi} K\sin(nx)\,dx\right].$$

$$= \frac{1}{\pi}\left[\frac{-K}{n}(-\cos nx)\Big|_{-\pi}^{0} + \frac{K}{n}\left(-\cos(nx)\right)\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[+\frac{K}{n}\left(\cos(0) - \cos(-n\pi)\right) - \frac{K}{n}\left(\cos(n\pi) - \cos(0)\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{K}{n}\left(1 - \cos(n\pi)\right) - \frac{K}{n}\left(\cos(n\pi) - 1\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{K}{n} - \frac{K}{n}\cos(n\pi) - \frac{K}{n}\cos(n\pi) + \frac{K}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{2K}{n} - \frac{2K}{n}\cos(n\pi)\right]$$

$$b_n = \frac{2K}{n\pi}\left[1 - \cos(n\pi)\right]$$

$\cos(-n\pi) = \cos(n\pi)$

n: even

$n = 1, 2, 3, \ldots$

$$\cos n\pi = \begin{cases} -1 & \text{for } n \text{ odd} \\ +1 & \text{for } n \text{ even} \end{cases}$$

فردي

زوجي

$$\therefore f(x) = A + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{2K}{n\pi}\left[1 - \cos(n\pi)\right]\sin(nx)$$

Ex: Find the Fourier Series for $f(x)=x$ in the interval $0<x<2\pi$.

Sol: $f(x)=A+\sum_{n=1}^{\infty} a_n \cos(nx)+\sum_{n=1}^{\infty} b_n \sin(nx)$

$P=2\pi$, half $P=\pi$

$\Rightarrow A=\frac{1}{2\pi}\int_0^{2\pi} f(x)\,.dx=\frac{1}{2\pi}\int_0^{2\pi} x\,.dx=\frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$

$A=\frac{1}{2\pi}\left[\frac{(2\pi)^2}{2}-0\right]=\frac{4\pi^2}{4\pi}=\pi$

$\Rightarrow a_n=\frac{1}{\pi}\int_0^{2\pi} x.\cos(nx).dx$

* by using By part integral method

$\int u\,dv=uv-\int v\,du$ $\rightsquigarrow$ let $u=x$, $dv=\cos nx.dx$

$du=dx$, $v=\frac{1}{n}\sin(nx)$

$\int_0^{2\pi} x\cos nx\,dx=\frac{x}{n}\sin nx-\int\frac{1}{n}\sin(nx).dx$

$=\left[\frac{x}{n}\sin nx+\frac{1}{n^2}\cos(nx)\right]_0^{2\pi}$

$\therefore a_n=\frac{1}{\pi}\left\{\left[\frac{x\sin(nx)}{n}\right]_0^{2\pi}+\left[\frac{\cos(nx)}{n^2}\right]_0^{2\pi}\right\}$

$a_n=\frac{1}{\pi}\left\{\left[\frac{(2\pi)\overset{0}{\cancel{\sin(2n\pi)}}}{n}-0\right]+\left[\frac{\overset{1\,(\text{even})}{\cancel{\cos(2n\pi)}}}{n^2}-\frac{\cos(0)}{n^2}\right]\right\}$

$a_n=\frac{1}{\pi}\left[(0-0)+\left(\frac{1}{n^2}-\frac{1}{n^2}\right)\right]\Rightarrow\boxed{a_n=0}$

$\Rightarrow b_n=\frac{1}{\pi}\int_0^{2\pi} x.\sin(nx).dx$

* let $u=x$, $dv=\sin(nx)\,dx$

$du=dx$, $v=\frac{-1}{n}\cos(nx)$

$\int u\,dv=uv-\int v\,du$

$$= \frac{-x}{n}\cos(nx) - \int \frac{-1}{n}\cos(nx)\,.dx$$

$$= \frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}$$

$$\therefore b_n = \frac{1}{\pi}\left\{\left[\frac{-x\cos(nx)}{n}\right]_0^{2\pi} + \left[\frac{\sin(nx)}{n^2}\right]_0^{2\pi}\right.$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{(-2\pi)\overset{1}{\cancel{\cos(2\pi n)}}}{n} + 0\right] + \left[\frac{\overset{0}{\cancel{\sin(2n\pi)}}}{\cancel{n^2}} - 0\right]\right\}$$

$$b_n = \frac{1}{\pi}\left[\frac{-2\pi(1)}{n} + 0 + 0 - 0\right] \Rightarrow \boxed{b_n = \frac{-2}{n}}$$

$$\therefore f(x) = A + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$f(x) = \pi + 0 + \sum_{n=1}^{\infty} \frac{-2}{n}\sin(nx)$$

$$f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

$$f(x) = \pi - 2\left\{\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots + \frac{\sin(nx)}{n}\right\}$$

Q/ Find Fourier Series of f

$$f(x) = \begin{cases} -\pi & \text{for } -\pi \leq x < 0 \\ x & \text{for } 0 \leq x \leq \pi \end{cases}$$

◆ Find the Fourier series associated with the function

$$f(x)=\begin{cases}-\pi & \text{for } -\pi \le x < 0,\\ x \quad , & \text{for } 0 \le x \le \pi.\end{cases}$$

Sol:

$$\mathbf{p} = \textit{higher - lower}$$

$$\mathbf{p} = \pi - (-\pi) = 2\pi$$

$$L = \frac{\mathbf{p}}{2} = \frac{2\pi}{2} = \pi$$

$$\because a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right\} = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\, dx + \int_{0}^{\pi} x dx\right\} =$$

$$= \frac{1}{\pi}\left[-\pi x \Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_{0}^{\pi}\right] = \frac{1}{\pi}\left[0 - \pi^2 + \frac{\pi^2}{2} - 0\right] = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx\, dx + \int_{0}^{\pi} \mathbf{x}\cos nx\, dx\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\frac{(-\pi)}{n}\int_{-\pi}^{0} n\cos nx\, dx + \int_{0}^{\pi} \mathbf{x}\cos nx\, dx\right]$$

let $u = x$ $\qquad \int dv = \int cosnxdx$

$du = dx$ $\qquad v = \frac{1}{n}\sin nx$

$$a_n = \frac{1}{\pi}\left[\frac{(-\pi)}{n}\int_{-\pi}^{0} n\cos nx\, dx + \frac{x}{n}\sin nx\Big|_{0}^{\pi} - \int_{0}^{\pi} \frac{n}{nn}\sin nx\, dx\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{-\pi}{n}\sin nx\Big|_{-\pi}^{0} + \frac{x}{n}\sin nx\Big|_{0}^{\pi} + \frac{1}{n^2}\cos nx\Big|_{0}^{\pi}\right]$$

$$a_n = \frac{1}{\pi}\left[0 + \frac{\pi}{n}\cancel{\sin(-nx)} + \frac{\pi}{n}\cancel{\sin n\pi} - 0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\overset{=1}{\cos 0}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2}(\cos n\pi - 1)\right] \qquad \cos n\pi = (-1)^n \begin{cases} 1 & \text{n=even} \\ -1 & \text{n=odd} \end{cases}$$

for n even $\quad a_n = \frac{1}{\pi}\left[\frac{1}{n^2}(\overset{1-1=0}{\cos n\pi - 1})\right] \rightarrow \boxed{a_n = 0}$

for n odd $\quad a_n = \frac{1}{\pi}\left[\frac{1}{n^2}(\overset{-1-1=-2}{\cos n\pi - 1})\right] \rightarrow \boxed{a_n = \frac{-2}{\pi n^2}}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

let $u = x \Rightarrow du = dx \qquad \int dv = \int \sin nx\,dx \Rightarrow v = \frac{-\cos nx}{n}$

$$\int_0^{\pi} x\sin nx\,dx = \frac{-x\cos nx dx}{n} - \int_0^{\pi}\frac{-\cos nx dx}{n} = \frac{-x\cos nx dx}{n} + \int_0^{\pi}\frac{n\cos nx dx}{n\,n} =$$

$$= \frac{-x\cos nx dx}{n} + \int_0^{\pi}\frac{n\cos nx dx}{n^2}$$

$$\therefore\ b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{-\pi n\sin nx\,dx}{n} - \frac{x\cos nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{n\cos nx dx}{n^2}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\pi}{n}\cos nx\Big|_{-\pi}^{0} - \frac{x\cos nx}{n}\Big|_0^{\pi} + \frac{1}{n^2}\sin nx\Big|_0^{\pi}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\pi}{n}\overset{1}{\cancel{\cos}}0 - \frac{\pi}{n}\overset{(-1)^n}{\cancel{\cos(-n\pi)}} - \left(\frac{\pi}{n}\overset{(-1)^n}{\cancel{\cos(n\pi)}} - \frac{0}{n}\cancel{\cos 0}\right) + \frac{1}{n^2}\cancel{\sin(n\pi)} - \cancel{\frac{1}{n^2}\sin 0}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\pi}{n}\left(1-(-1)^n-(-1)^n\right)\right]$$

for n even

$$b_n = \frac{1}{\pi}\left[\frac{\pi}{n}\left(1-1-1\right)\right] = \frac{1}{\cancel{\pi}}\left(-\frac{\cancel{\pi}}{n}\right) = \boxed{\frac{-1}{n}}$$

for n odd

$$b_n = \frac{1}{\pi}\left[\frac{\pi}{n}\left(1+1+1\right)\right] = \frac{1}{\cancel{\pi}}\left(\frac{3\cancel{\pi}}{n}\right) = \boxed{\frac{3}{n}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\cancel{\pi}}{\cancel{L}}^{\,L=\pi} x + b_n \sin\frac{n\cancel{\pi}}{\cancel{L}} x$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{-2}{\pi n^2}\cos nx + \frac{3}{n}\sin nx\right)$$

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{-2}{\pi}\frac{\cos nx}{n^2} + 3\,\frac{\sin nx}{n}\right)$$

for odd n

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{-2}{\pi}\cos x + 3\sin x + \frac{-2}{\pi}\frac{\cos 3x}{9} + 3\,\frac{\sin 3x}{3} + \dots\right)$$

5……7…9…i.c