

2.4 Addition of a System of Coplanar Forces

When a force is resolved into two components along the x and y axes, the components are then called **rectangular components**. For analytical work we can represent these components in one of two ways, using either scalar or Cartesian vector notation.

Scalar Notation. The rectangular components of force $\mathbf{F}$ shown in Fig. 2-15a are found using the parallelogram law, so that $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$. Because these components form a right triangle, they can be determined from

$$F_x = F \cos \theta \quad \text{and} \quad F_y = F \sin \theta$$

Instead of using the angle θ , however, the direction of $\mathbf{F}$ can also be defined using a small “slope” triangle, as in the example shown in Fig. 2-15b. Since this triangle and the larger shaded triangle are similar, the proportional length of the sides gives

$$\frac{F_x}{F} = \frac{a}{c}$$

or

$$F_x = F \left(\frac{a}{c} \right)$$

and

$$\frac{F_y}{F} = \frac{b}{c}$$

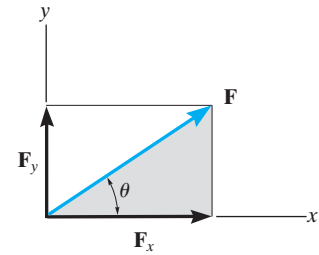
or

$$F_y = -F \left(\frac{b}{c} \right)$$

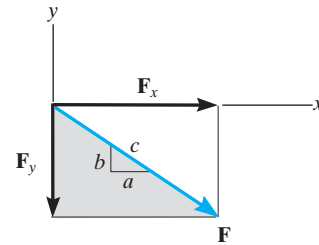
Here the y component is a *negative scalar* since $\mathbf{F}_y$ is directed along the negative y axis.

It is important to keep in mind that this positive and negative scalar notation is to be used only for computational purposes, not for graphical representations in figures. Throughout the book, the *head of a vector arrow in any figure* indicates the sense of the vector *graphically*; algebraic signs are not used for this purpose. Thus, the vectors in Figs. 2-15a and 2-15b are designated by using boldface (vector) notation.* Whenever italic symbols are written near vector arrows in figures, they indicate the *magnitude* of the vector, which is *always a positive* quantity.

*Negative signs are used only in figures with boldface notation when showing equal but opposite pairs of vectors, as in Fig. 2-2.



(a)



(b)

Fig. 2-15

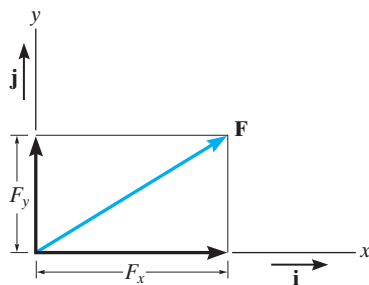


Fig. 2-16

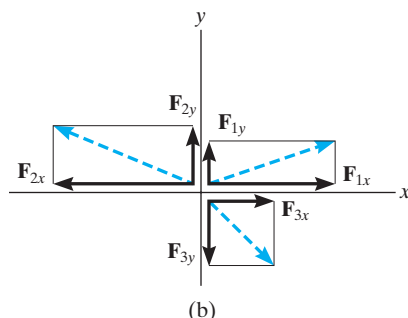
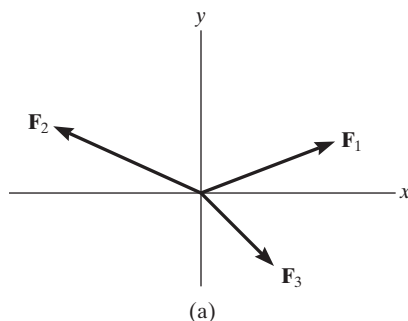
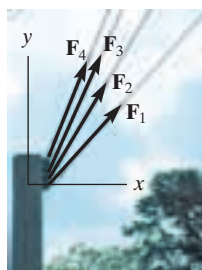


Fig. 2-17



The resultant force of the four cable forces acting on the post can be determined by adding algebraically the separate x and y components of each cable force. This resultant $\mathbf{F}_R$ produces the *same pulling effect* on the post as all four cables. (© Russell C. Hibbeler)

Cartesian Vector Notation. It is also possible to represent the x and y components of a force in terms of Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$. They are called unit vectors because they have a dimensionless magnitude of 1, and so they can be used to designate the *directions* of the x and y axes, respectively, Fig. 2-16.*

Since the *magnitude* of each component of $\mathbf{F}$ is *always a positive quantity*, which is represented by the (positive) scalars F_x and F_y , then we can express $\mathbf{F}$ as a **Cartesian vector**,

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j}$$

Coplanar Force Resultants. We can use either of the two methods just described to determine the resultant of several **coplanar forces**, i.e., forces that all lie in the same plane. To do this, each force is first resolved into its x and y components, and then the respective components are added using *scalar algebra* since they are collinear. The resultant force is then formed by adding the resultant components using the parallelogram law. For example, consider the three concurrent forces in Fig. 2-17a, which have x and y components shown in Fig. 2-17b. Using Cartesian vector notation, each force is first represented as a Cartesian vector, i.e.,

$$\mathbf{F}_1 = F_{1x} \mathbf{i} + F_{1y} \mathbf{j}$$

$$\mathbf{F}_2 = -F_{2x} \mathbf{i} + F_{2y} \mathbf{j}$$

$$\mathbf{F}_3 = F_{3x} \mathbf{i} - F_{3y} \mathbf{j}$$

The vector resultant is therefore

$$\begin{aligned} \mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\ &= F_{1x} \mathbf{i} + F_{1y} \mathbf{j} - F_{2x} \mathbf{i} + F_{2y} \mathbf{j} + F_{3x} \mathbf{i} - F_{3y} \mathbf{j} \\ &= (F_{1x} - F_{2x} + F_{3x}) \mathbf{i} + (F_{1y} + F_{2y} - F_{3y}) \mathbf{j} \\ &= (F_{Rx}) \mathbf{i} + (F_{Ry}) \mathbf{j} \end{aligned}$$

If *scalar notation* is used, then indicating the positive directions of components along the x and y axes with symbolic arrows, we have

$$\rightarrow \quad (F_R)_x = F_{1x} - F_{2x} + F_{3x}$$

$$+ \uparrow \quad (F_R)_y = F_{1y} + F_{2y} - F_{3y}$$

These are the *same* results as the $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{F}_R$ determined above.

*For handwritten work, unit vectors are usually indicated using a circumflex, e.g., $\hat{i}$ and $\hat{j}$. Also, realize that F_x and F_y in Fig. 2-16 represent the *magnitudes* of the components, which are *always positive scalars*. The directions are defined by $\mathbf{i}$ and $\mathbf{j}$. If instead we used scalar notation, then F_x and F_y could be positive or negative scalars, since they would account for *both* the magnitude and direction of the components.

We can represent the components of the resultant force of any number of coplanar forces symbolically by the algebraic sum of the x and y components of all the forces, i.e.,

$$\begin{aligned} (F_R)_x &= \Sigma F_x \\ (F_R)_y &= \Sigma F_y \end{aligned} \quad (2-1)$$

Once these components are determined, they may be sketched along the x and y axes with their proper sense of direction, and the resultant force can be determined from vector addition, as shown in Fig. 2-17c. From this sketch, the magnitude of $\mathbf{F}_R$ is then found from the Pythagorean theorem; that is,

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2}$$

Also, the angle θ , which specifies the direction of the resultant force, is determined from trigonometry:

$$\theta = \tan^{-1} \left| \frac{(F_R)_y}{(F_R)_x} \right|$$

The above concepts are illustrated numerically in the examples which follow.

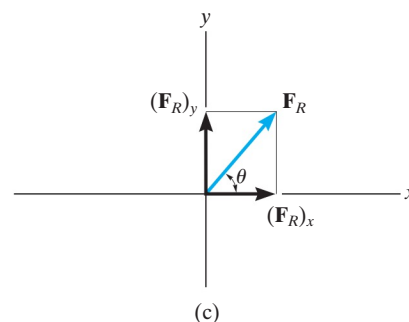
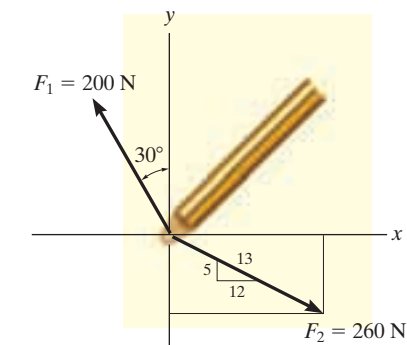


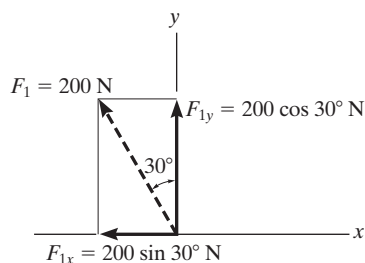
Fig. 2-17 (cont.)

Important Points

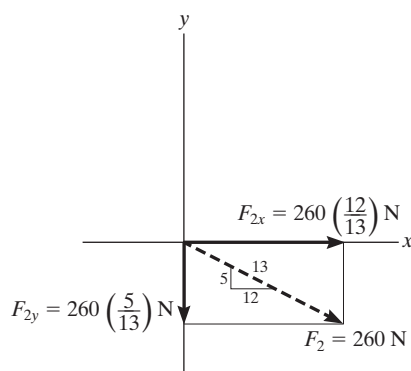
- The resultant of several coplanar forces can easily be determined if an x, y coordinate system is established and the forces are resolved along the axes.
- The direction of each force is specified by the angle its line of action makes with one of the axes, or by a slope triangle.
- The orientation of the x and y axes is arbitrary, and their positive direction can be specified by the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$.
- The x and y components of the *resultant force* are simply the algebraic addition of the components of all the coplanar forces.
- The magnitude of the resultant force is determined from the Pythagorean theorem, and when the resultant components are sketched on the x and y axes, Fig. 2-17c, the direction θ can be determined from trigonometry.



(a)



(b)



(c)

Fig. 2-18

Determine the x and y components of $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the boom shown in Fig. 2-18a. Express each force as a Cartesian vector.

SOLUTION

Scalar Notation. By the parallelogram law, $\mathbf{F}_1$ is resolved into x and y components, Fig. 2-18b. Since $\mathbf{F}_{1x}$ acts in the $-x$ direction, and $\mathbf{F}_{1y}$ acts in the $+y$ direction, we have

$$F_{1x} = -200 \sin 30^\circ \text{ N} = -100 \text{ N} = 100 \text{ N} \leftarrow \quad \text{Ans.}$$

$$F_{1y} = 200 \cos 30^\circ \text{ N} = 173 \text{ N} = 173 \text{ N} \uparrow \quad \text{Ans.}$$

The force $\mathbf{F}_2$ is resolved into its x and y components, as shown in Fig. 2-18c. Here the *slope* of the line of action for the force is indicated. From this “slope triangle” we could obtain the angle θ , e.g., $\theta = \tan^{-1}\left(\frac{5}{12}\right)$, and then proceed to determine the magnitudes of the components in the same manner as for $\mathbf{F}_1$. The easier method, however, consists of using proportional parts of similar triangles, i.e.,

$$\frac{F_{2x}}{260 \text{ N}} = \frac{12}{13} \quad F_{2x} = 260 \text{ N} \left(\frac{12}{13} \right) = 240 \text{ N}$$

Similarly,

$$F_{2y} = 260 \text{ N} \left(\frac{5}{13} \right) = 100 \text{ N}$$

Notice how the magnitude of the *horizontal component*, $\mathbf{F}_{2x}$, was obtained by multiplying the force magnitude by the ratio of the *horizontal leg* of the slope triangle divided by the hypotenuse; whereas the magnitude of the *vertical component*, $\mathbf{F}_{2y}$, was obtained by multiplying the force magnitude by the ratio of the *vertical leg* divided by the hypotenuse. Hence, using scalar notation to represent these components, we have

$$F_{2x} = 240 \text{ N} = 240 \text{ N} \rightarrow \quad \text{Ans.}$$

$$F_{2y} = -100 \text{ N} = 100 \text{ N} \downarrow \quad \text{Ans.}$$

Cartesian Vector Notation. Having determined the magnitudes and directions of the components of each force, we can express each force as a Cartesian vector.

$$\mathbf{F}_1 = \{-100\mathbf{i} + 173\mathbf{j}\} \text{ N} \quad \text{Ans.}$$

$$\mathbf{F}_2 = \{240\mathbf{i} - 100\mathbf{j}\} \text{ N} \quad \text{Ans.}$$

EXAMPLE 2.6

The link in Fig. 2–19a is subjected to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and direction of the resultant force.

SOLUTION I

Scalar Notation. First we resolve each force into its x and y components, Fig. 2–19b, then we sum these components algebraically.

$$\begin{aligned} \rightarrow (F_R)_x = \Sigma F_x; \quad (F_R)_x &= 600 \cos 30^\circ \text{ N} - 400 \sin 45^\circ \text{ N} \\ &= 236.8 \text{ N} \rightarrow \end{aligned}$$

$$\begin{aligned} +\uparrow (F_R)_y = \Sigma F_y; \quad (F_R)_y &= 600 \sin 30^\circ \text{ N} + 400 \cos 45^\circ \text{ N} \\ &= 582.8 \text{ N} \uparrow \end{aligned}$$

The resultant force, shown in Fig. 2–19c, has a *magnitude* of

$$\begin{aligned} F_R &= \sqrt{(236.8 \text{ N})^2 + (582.8 \text{ N})^2} \\ &= 629 \text{ N} \end{aligned}$$

Ans.

From the vector addition,

$$\theta = \tan^{-1} \left(\frac{582.8 \text{ N}}{236.8 \text{ N}} \right) = 67.9^\circ$$

*Ans.***SOLUTION II**

Cartesian Vector Notation. From Fig. 2–19b, each force is first expressed as a Cartesian vector.

$$\mathbf{F}_1 = \{600 \cos 30^\circ \mathbf{i} + 600 \sin 30^\circ \mathbf{j}\} \text{ N}$$

$$\mathbf{F}_2 = \{-400 \sin 45^\circ \mathbf{i} + 400 \cos 45^\circ \mathbf{j}\} \text{ N}$$

Then,

$$\begin{aligned} \mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 = (600 \cos 30^\circ \text{ N} - 400 \sin 45^\circ \text{ N})\mathbf{i} \\ &\quad + (600 \sin 30^\circ \text{ N} + 400 \cos 45^\circ \text{ N})\mathbf{j} \\ &= \{236.8\mathbf{i} + 582.8\mathbf{j}\} \text{ N} \end{aligned}$$

The magnitude and direction of $\mathbf{F}_R$ are determined in the same manner as before.

NOTE: Comparing the two methods of solution, notice that the use of scalar notation is more efficient since the components can be found *directly*, without first having to express each force as a Cartesian vector before adding the components. Later, however, we will show that Cartesian vector analysis is very beneficial for solving three-dimensional problems.

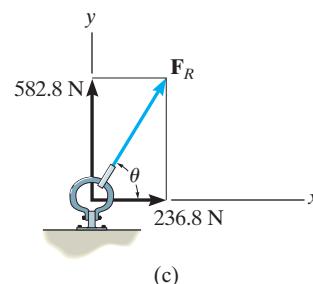
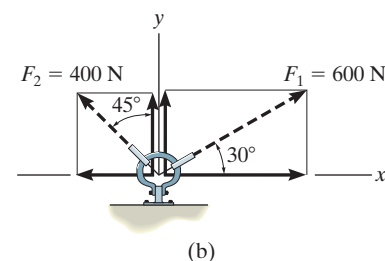
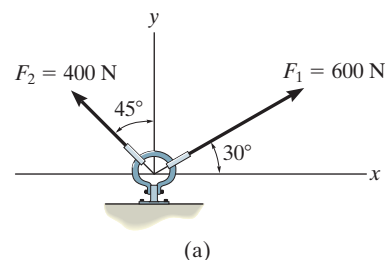


Fig. 2–19

The end of the boom O in Fig. 2–20*a* is subjected to three concurrent and coplanar forces. Determine the magnitude and direction of the resultant force.

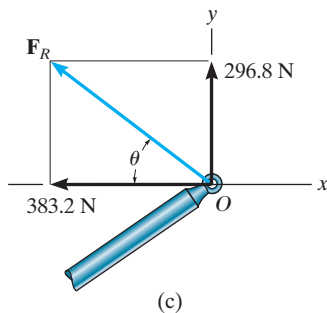
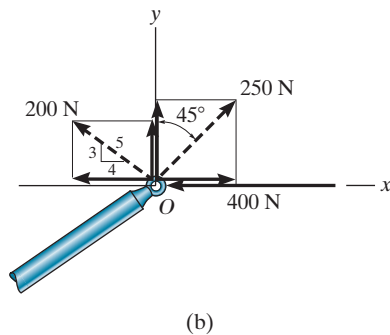
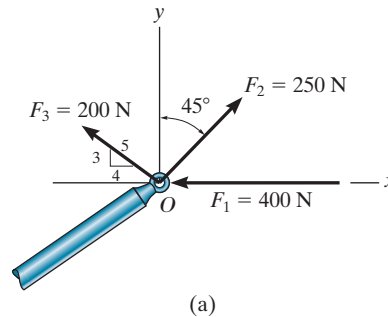


Fig. 2–20

SOLUTION

Each force is resolved into its x and y components, Fig. 2–20*b*. Summing the x components, we have

$$\begin{aligned} \rightarrow (F_R)_x &= \Sigma F_x; & (F_R)_x &= -400 \text{ N} + 250 \sin 45^\circ \text{ N} - 200\left(\frac{4}{5}\right) \text{ N} \\ & & &= -383.2 \text{ N} = 383.2 \text{ N} \leftarrow \end{aligned}$$

The negative sign indicates that F_{Rx} acts to the left, i.e., in the negative x direction, as noted by the small arrow. Obviously, this occurs because F_1 and F_3 in Fig. 2–20*b* contribute a greater pull to the left than F_2 which pulls to the right. Summing the y components yields

$$\begin{aligned} + \uparrow (F_R)_y &= \Sigma F_y; & (F_R)_y &= 250 \cos 45^\circ \text{ N} + 200\left(\frac{3}{5}\right) \text{ N} \\ & & &= 296.8 \text{ N} \uparrow \end{aligned}$$

The resultant force, shown in Fig. 2–20*c*, has a *magnitude* of

$$\begin{aligned} F_R &= \sqrt{(-383.2 \text{ N})^2 + (296.8 \text{ N})^2} \\ &= 485 \end{aligned} \quad \text{Ans.}$$

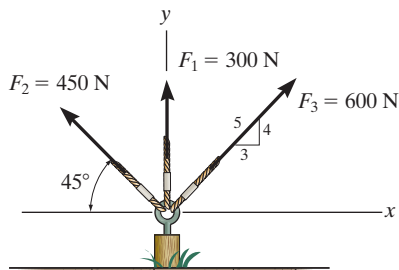
From the vector addition in Fig. 2–20*c*, the direction angle θ is

$$\theta = \tan^{-1}\left(\frac{296.8}{383.2}\right) = 37.8^\circ \quad \text{Ans.}$$

NOTE: Application of this method is more convenient, compared to using two applications of the parallelogram law, first to add $\mathbf{F}_1$ and $\mathbf{F}_2$ then adding $\mathbf{F}_3$ to this resultant.

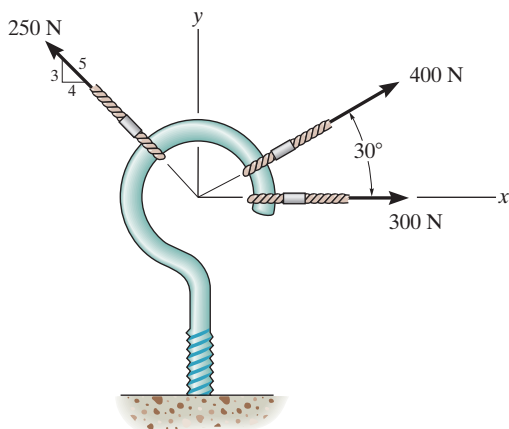
FUNDAMENTAL PROBLEMS

F2-7. Resolve each force acting on the post into its x and y components.



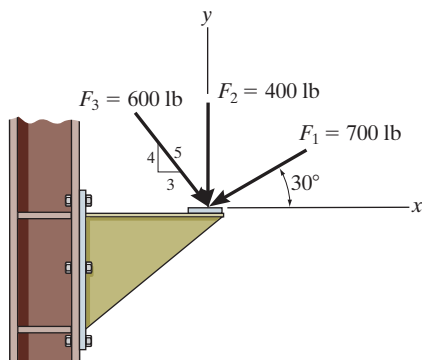
Prob. F2-7

F2-8. Determine the magnitude and direction of the resultant force.



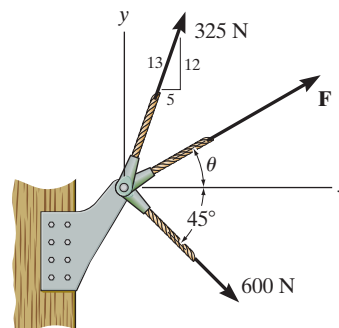
Prob. F2-8

F2-9. Determine the magnitude of the resultant force acting on the corbel and its direction θ measured counterclockwise from the x axis.



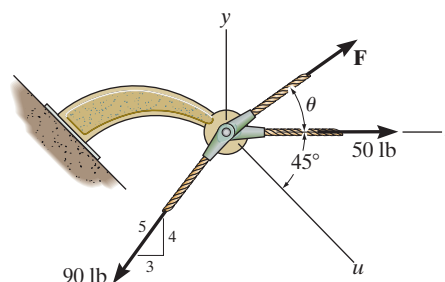
Prob. F2-9

F2-10. If the resultant force acting on the bracket is to be 750 N directed along the positive x axis, determine the magnitude of $\mathbf{F}$ and its direction θ .



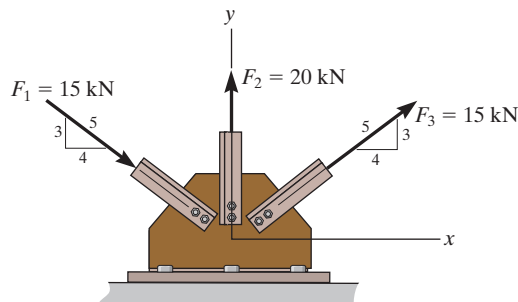
Prob. F2-10

F2-11. If the magnitude of the resultant force acting on the bracket is to be 80 lb directed along the u axis, determine the magnitude of $\mathbf{F}$ and its direction θ .



Prob. F2-11

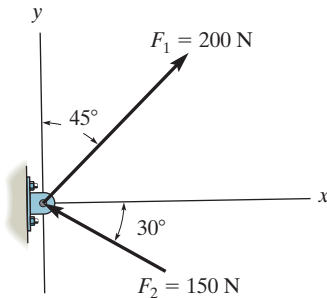
F2-12. Determine the magnitude of the resultant force and its direction θ measured counterclockwise from the positive x axis.



Prob. F2-12

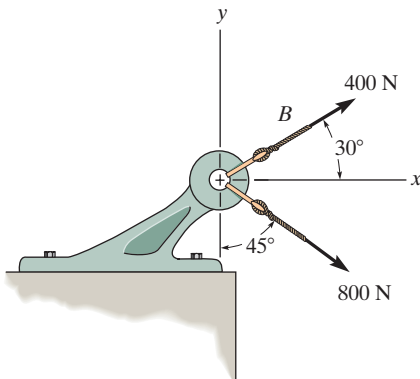
PROBLEMS

***2–32.** Determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive x axis.



Prob. 2–32

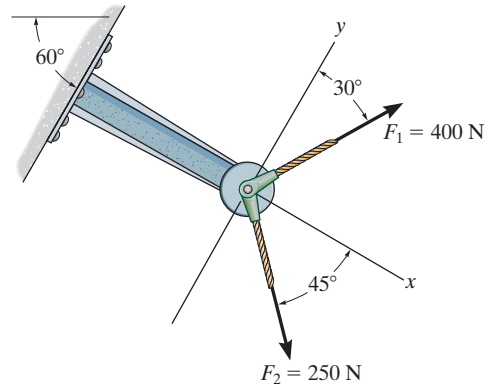
2–33. Determine the magnitude of the resultant force and its direction, measured clockwise from the positive x axis.



Prob. 2–33

2–34. Resolve $\mathbf{F}_1$ and $\mathbf{F}_2$ into their x and y components.

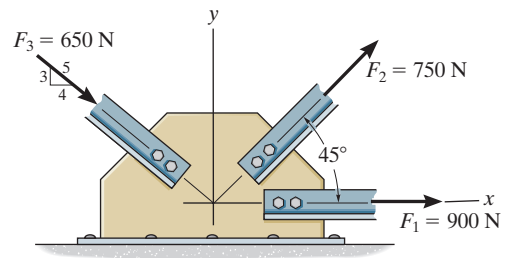
2–35. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.



Probs. 2–34/35

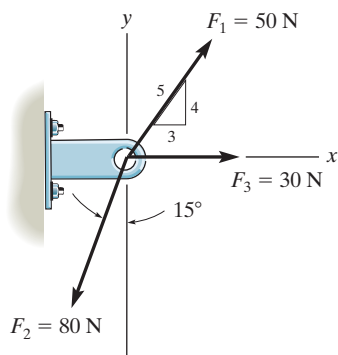
***2–36.** Resolve each force acting on the *gusset plate* into its x and y components, and express each force as a Cartesian vector.

2–37. Determine the magnitude of the resultant force acting on the plate and its direction, measured counterclockwise from the positive x axis.



Probs. 2–36/37

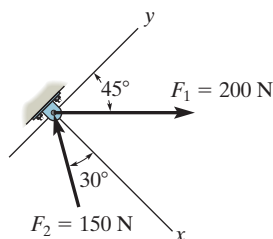
2-38. Express each of the three forces acting on the support in Cartesian vector form and determine the magnitude of the resultant force and its direction, measured clockwise from positive x axis.



Prob. 2-38

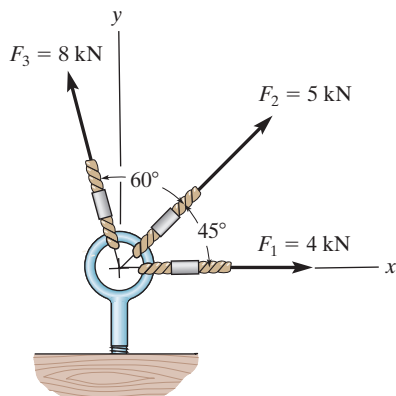
2-39. Determine the x and y components of $\mathbf{F}_1$ and $\mathbf{F}_2$.

***2-40.** Determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive x axis.



Probs. 2-39/40

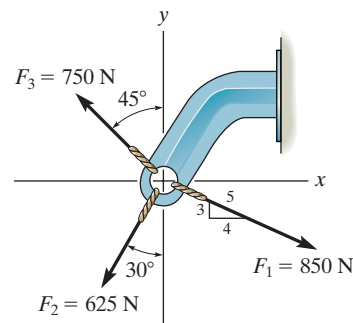
2-41. Determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive x axis.



Prob. 2-41

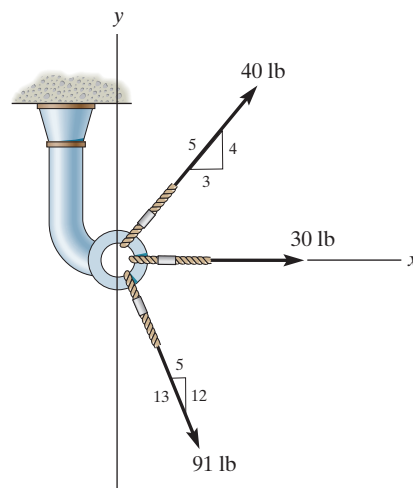
2-42. Express $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ as Cartesian vectors.

2-43. Determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive x axis.



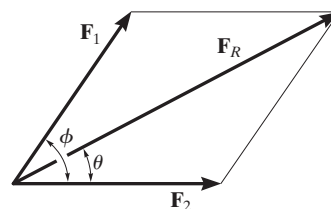
Probs. 2-42/43

***2-44.** Determine the magnitude of the resultant force and its direction, measured clockwise from the positive x axis.



Prob. 2-44

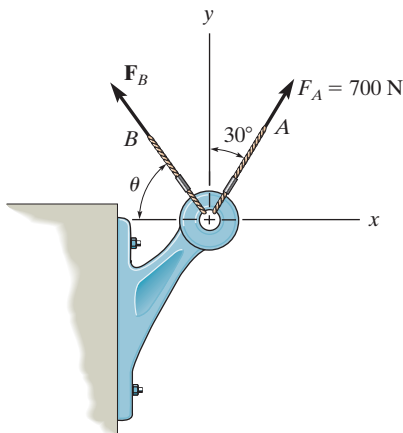
2-45. Determine the magnitude and direction θ of the resultant force $\mathbf{F}_R$. Express the result in terms of the magnitudes of the components $\mathbf{F}_1$ and $\mathbf{F}_2$ and the angle ϕ .



Prob. 2-45

2-46. Determine the magnitude and orientation θ of $\mathbf{F}_B$ so that the resultant force is directed along the positive y axis and has a magnitude of 1500 N.

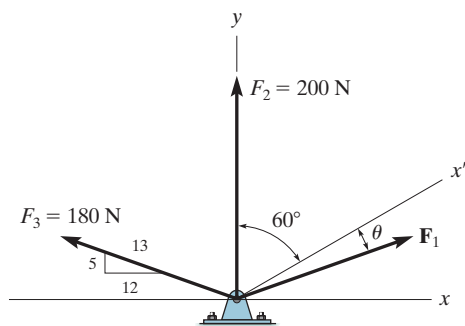
2-47. Determine the magnitude and orientation, measured counterclockwise from the positive y axis, of the resultant force acting on the bracket, if $F_B = 600$ N and $\theta = 20^\circ$.



Probs. 2-46/47

***2-48.** Three forces act on the bracket. Determine the magnitude and direction θ of $\mathbf{F}_1$ so that the resultant force is directed along the positive x' axis and has a magnitude of 800 N.

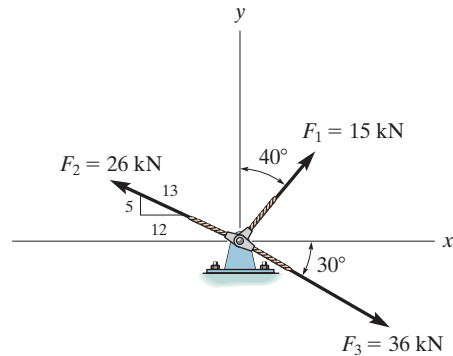
2-49. If $F_1 = 300$ N and $\theta = 10^\circ$, determine the magnitude and direction, measured counterclockwise from the positive x' axis, of the resultant force acting on the bracket.



Probs. 2-48/49

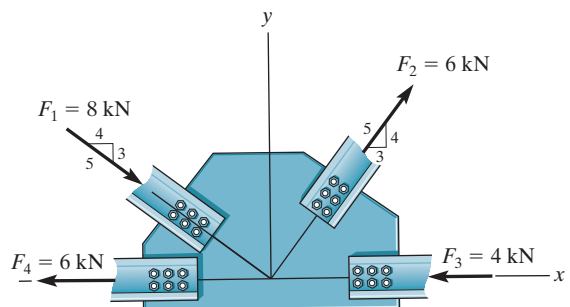
2-50. Express $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ as Cartesian vectors.

2-51. Determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive x axis.



Probs. 2-50/51

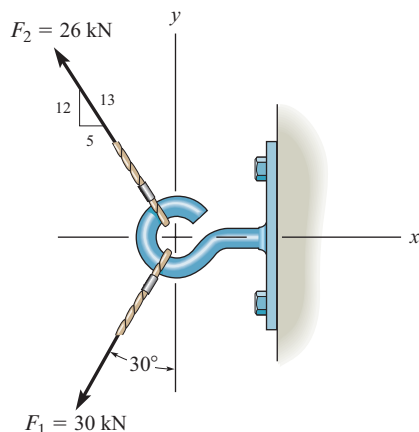
***2-52.** Determine the x and y components of each force acting on the *gusset plate* of a bridge truss. Show that the resultant force is zero.



Prob. 2-52

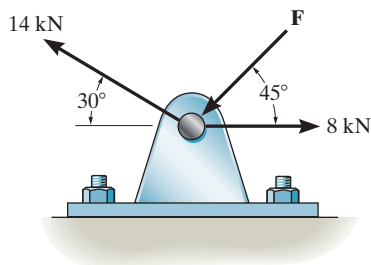
2-53. Express $\mathbf{F}_1$ and $\mathbf{F}_2$ as Cartesian vectors.

2-54. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.



Probs. 2-53/54

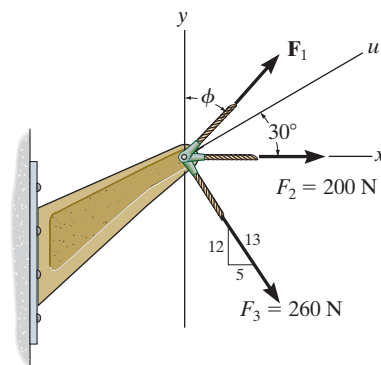
2-55. Determine the magnitude of force $\mathbf{F}$ so that the resultant force of the three forces is as small as possible. What is the magnitude of the resultant force?



Prob. 2-55

***2-56.** If the magnitude of the resultant force acting on the bracket is to be 450 N directed along the positive u axis, determine the magnitude of $\mathbf{F}_1$ and its direction ϕ .

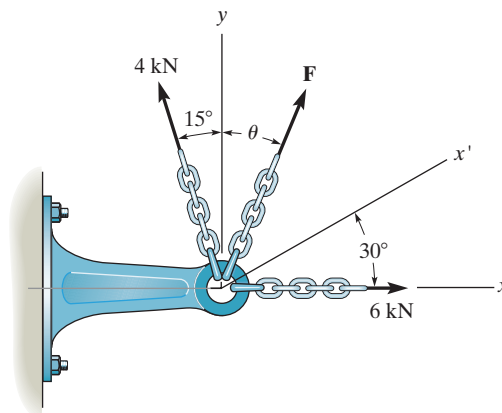
2-57. If the resultant force acting on the bracket is required to be a minimum, determine the magnitudes of $\mathbf{F}_1$ and the resultant force. Set $\phi = 30^\circ$.



Probs. 2-56/57

2-58. Three forces act on the bracket. Determine the magnitude and direction θ of $\mathbf{F}$ so that the resultant force is directed along the positive x' axis and has a magnitude of 8 kN.

2-59. If $F = 5$ kN and $\theta = 30^\circ$, determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive x axis.



Probs. 2-58/59

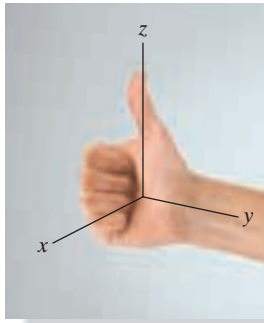


Fig. 2-21 (© Russell C. Hibbeler)

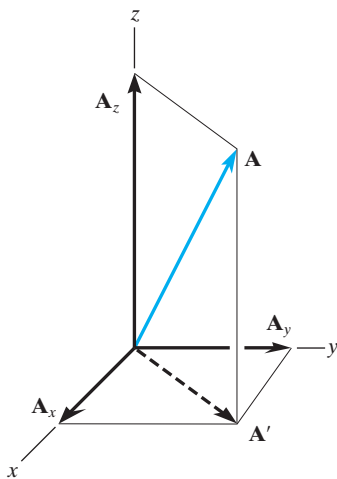


Fig. 2-22

2.5 Cartesian Vectors

The operations of vector algebra, when applied to solving problems in *three dimensions*, are greatly simplified if the vectors are first represented in Cartesian vector form. In this section we will present a general method for doing this; then in the next section we will use this method for finding the resultant force of a system of concurrent forces.

Right-Handed Coordinate System. We will use a right-handed coordinate system to develop the theory of vector algebra that follows. A rectangular coordinate system is said to be **right-handed** if the thumb of the right hand points in the direction of the positive z axis when the right-hand fingers are curled about this axis and directed from the positive x towards the positive y axis, Fig. 2-21.

Rectangular Components of a Vector. A vector $\mathbf{A}$ may have one, two, or three rectangular components along the x , y , z coordinate axes, depending on how the vector is oriented relative to the axes. In general, though, when $\mathbf{A}$ is directed within an octant of the x , y , z frame, Fig. 2-22, then by two successive applications of the parallelogram law, we may resolve the vector into components as $\mathbf{A} = \mathbf{A}' + \mathbf{A}_z$ and then $\mathbf{A}' = \mathbf{A}_x + \mathbf{A}_y$. Combining these equations, to eliminate $\mathbf{A}'$, $\mathbf{A}$ is represented by the vector sum of its *three* rectangular components,

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z \quad (2-2)$$

Cartesian Unit Vectors. In three dimensions, the set of Cartesian unit vectors, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, is used to designate the directions of the x , y , z axes, respectively. As stated in Sec. 2-4, the *sense* (or arrowhead) of these vectors will be represented analytically by a plus or minus sign, depending on whether they are directed along the positive or negative x , y , or z axes. The positive Cartesian unit vectors are shown in Fig. 2-23.

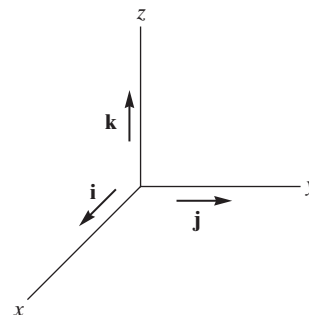


Fig. 2-23

Cartesian Vector Representation. Since the three components of $\mathbf{A}$ in Eq. 2-2 act in the positive $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ directions, Fig. 2-24, we can write $\mathbf{A}$ in Cartesian vector form as

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \quad (2-3)$$

There is a distinct advantage to writing vectors in this manner. Separating the *magnitude* and *direction* of each *component vector* will simplify the operations of vector algebra, particularly in three dimensions.

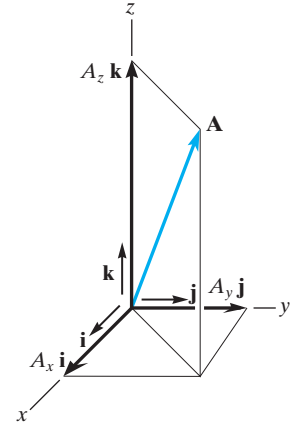


Fig. 2-24

Magnitude of a Cartesian Vector. It is always possible to obtain the magnitude of $\mathbf{A}$ provided it is expressed in Cartesian vector form. As shown in Fig. 2-25, from the blue right triangle, $A = \sqrt{A'^2 + A_z^2}$, and from the gray right triangle, $A' = \sqrt{A_x^2 + A_y^2}$. Combining these equations to eliminate A' yields

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (2-4)$$

Hence, the magnitude of $\mathbf{A}$ is equal to the positive square root of the sum of the squares of its components.

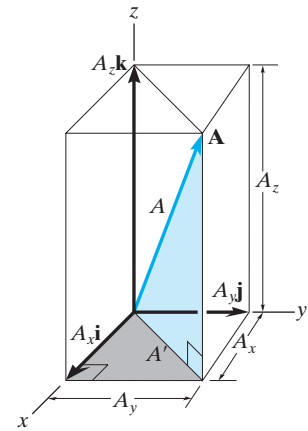


Fig. 2-25

Coordinate Direction Angles. We will define the *direction* of $\mathbf{A}$ by the **coordinate direction angles** α (alpha), β (beta), and γ (gamma), measured between the *tail* of $\mathbf{A}$ and the *positive* x , y , z axes provided they are located at the tail of $\mathbf{A}$, Fig. 2-26. Note that regardless of where $\mathbf{A}$ is directed, each of these angles will be between 0° and 180° .

To determine α , β , and γ , consider the projection of $\mathbf{A}$ onto the x , y , z axes, Fig. 2-27. Referring to the colored right triangles shown in the figure, we have

$$\cos \alpha = \frac{A_x}{A} \quad \cos \beta = \frac{A_y}{A} \quad \cos \gamma = \frac{A_z}{A} \quad (2-5)$$

These numbers are known as the **direction cosines** of $\mathbf{A}$. Once they have been obtained, the coordinate direction angles α , β , γ can then be determined from the inverse cosines.

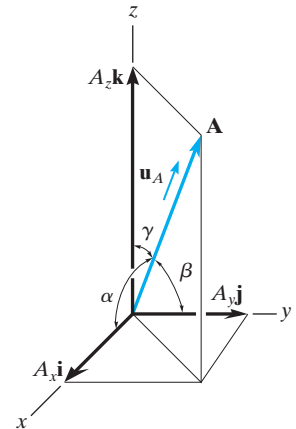


Fig. 2-26

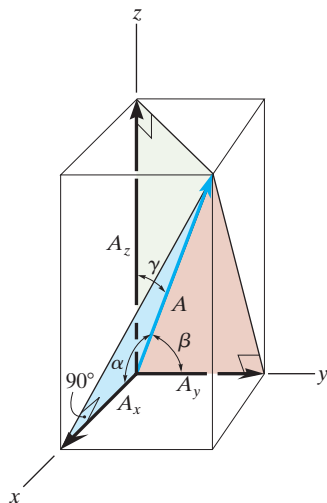


Fig. 2-27

An easy way of obtaining these direction cosines is to form a unit vector $\mathbf{u}_A$ in the direction of $\mathbf{A}$, Fig. 2-26. If $\mathbf{A}$ is expressed in Cartesian vector form, $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, then $\mathbf{u}_A$ will have a magnitude of one and be dimensionless provided $\mathbf{A}$ is divided by its magnitude, i.e.,

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k} \quad (2-6)$$

where $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$. By comparison with Eqs. 2-5, it is seen that the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of $\mathbf{u}_A$ represent the direction cosines of $\mathbf{A}$, i.e.,

$$\mathbf{u}_A = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k} \quad (2-7)$$

Since the magnitude of a vector is equal to the positive square root of the sum of the squares of the magnitudes of its components, and $\mathbf{u}_A$ has a magnitude of one, then from the above equation an important relation among the direction cosines can be formulated as

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad (2-8)$$

Here we can see that if only *two* of the coordinate angles are known, the third angle can be found using this equation.

Finally, if the magnitude and coordinate direction angles of $\mathbf{A}$ are known, then $\mathbf{A}$ may be expressed in Cartesian vector form as

$$\begin{aligned} \mathbf{A} &= A \mathbf{u}_A \\ &= A \cos \alpha \mathbf{i} + A \cos \beta \mathbf{j} + A \cos \gamma \mathbf{k} \\ &= A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \end{aligned} \quad (2-9)$$

Transverse and Azimuth Angles. Sometimes, the direction of $\mathbf{A}$ can be specified using two angles, namely, a **transverse angle** θ and an **azimuth angle** ϕ (phi), such as shown in Fig. 2-28. The components of $\mathbf{A}$ can then be determined by applying trigonometry first to the light blue right triangle, which yields

$$A_z = A \cos \phi$$

and

$$A' = A \sin \phi$$

Now applying trigonometry to the dark blue right triangle,

$$A_x = A' \cos \theta = A \sin \phi \cos \theta$$

$$A_y = A' \sin \theta = A \sin \phi \sin \theta$$

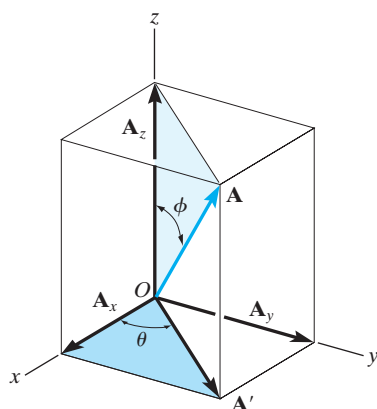


Fig. 2-28

Therefore $\mathbf{A}$ written in Cartesian vector form becomes

$$\mathbf{A} = A \sin \phi \cos \theta \mathbf{i} + A \sin \phi \sin \theta \mathbf{j} + A \cos \phi \mathbf{k}$$

You should not memorize this equation, rather it is important to understand how the components were determined using trigonometry.

2.6 Addition of Cartesian Vectors

The addition (or subtraction) of two or more vectors is greatly simplified if the vectors are expressed in terms of their Cartesian components. For example, if $\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$ and $\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$, Fig. 2–29, then the resultant vector, $\mathbf{R}$, has components which are the scalar sums of the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of $\mathbf{A}$ and $\mathbf{B}$, i.e.,

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

If this is generalized and applied to a system of several concurrent forces, then the force resultant is the vector sum of all the forces in the system and can be written as

$$\mathbf{F}_R = \Sigma \mathbf{F} = \Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k} \quad (2-10)$$

Here ΣF_x , ΣF_y , and ΣF_z represent the algebraic sums of the respective x, y, z or $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of each force in the system.

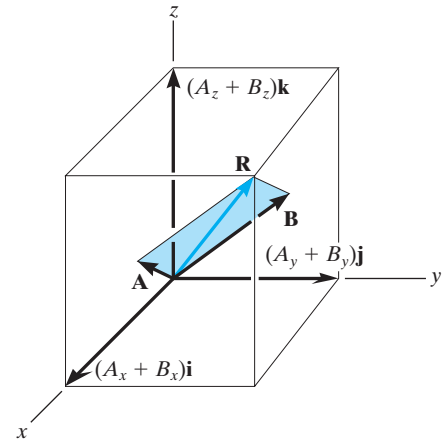


Fig. 2–29



Cartesian vector analysis provides a convenient method for finding both the resultant force and its components in three dimensions. (© Russell C. Hibbeler)

Important Points

- A Cartesian vector $\mathbf{A}$ has $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components along the x, y, z axes. If $\mathbf{A}$ is known, its magnitude is defined by $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.
- The direction of a Cartesian vector can be defined by the three angles α, β, γ , measured from the *positive* x, y, z axes to the *tail* of the vector. To find these angles formulate a unit vector in the direction of $\mathbf{A}$, i.e., $\mathbf{u}_A = \mathbf{A}/A$, and determine the inverse cosines of its components. Only two of these angles are independent of one another; the third angle is found from $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.
- The direction of a Cartesian vector can also be specified using a transverse angle θ and azimuth angle ϕ .

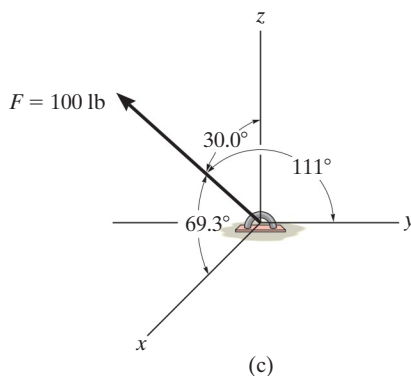
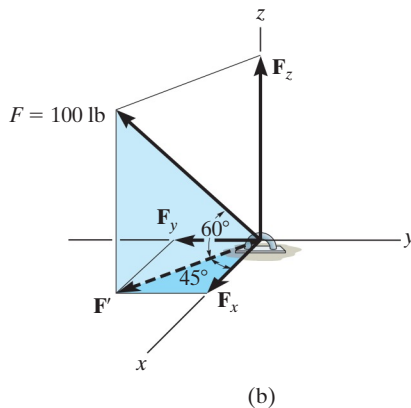
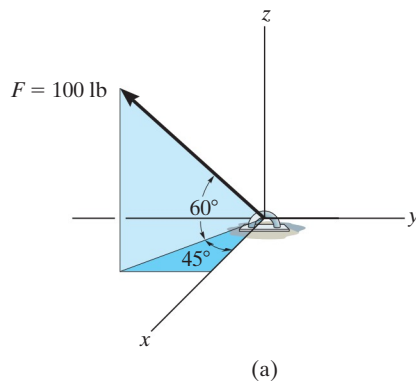


Fig. 2-30

Express the force $\mathbf{F}$ shown in Fig. 2-30a as a Cartesian vector.

SOLUTION

The angles of 60° and 45° defining the direction of $\mathbf{F}$ are *not* coordinate direction angles. Two successive applications of the parallelogram law are needed to resolve $\mathbf{F}$ into its x , y , z components. First $\mathbf{F} = \mathbf{F}' + \mathbf{F}_z$, then $\mathbf{F}' = \mathbf{F}_x + \mathbf{F}_y$, Fig. 2-30b. By trigonometry, the magnitudes of the components are

$$F_z = 100 \sin 60^\circ \text{ lb} = 86.6 \text{ lb}$$

$$F' = 100 \cos 60^\circ \text{ lb} = 50 \text{ lb}$$

$$F_x = F' \cos 45^\circ = 50 \cos 45^\circ \text{ lb} = 35.4 \text{ lb}$$

$$F_y = F' \sin 45^\circ = 50 \sin 45^\circ \text{ lb} = 35.4 \text{ lb}$$

Realizing that $\mathbf{F}_y$ has a direction defined by $-\mathbf{j}$, we have

$$\mathbf{F} = \{35.4\mathbf{i} - 35.4\mathbf{j} + 86.6\mathbf{k}\} \text{ lb}$$

Ans.

To show that the magnitude of this vector is indeed 100 lb, apply Eq. 2-4,

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2 + F_z^2} \\ &= \sqrt{(35.4)^2 + (35.4)^2 + (86.6)^2} = 100 \text{ lb} \end{aligned}$$

If needed, the coordinate direction angles of $\mathbf{F}$ can be determined from the components of the unit vector acting in the direction of $\mathbf{F}$. Hence,

$$\begin{aligned} \mathbf{u} &= \frac{\mathbf{F}}{F} = \frac{F_x}{F}\mathbf{i} + \frac{F_y}{F}\mathbf{j} + \frac{F_z}{F}\mathbf{k} \\ &= \frac{35.4}{100}\mathbf{i} - \frac{35.4}{100}\mathbf{j} + \frac{86.6}{100}\mathbf{k} \\ &= 0.354\mathbf{i} - 0.354\mathbf{j} + 0.866\mathbf{k} \end{aligned}$$

so that

$$\alpha = \cos^{-1}(0.354) = 69.3^\circ$$

$$\beta = \cos^{-1}(-0.354) = 111^\circ$$

$$\gamma = \cos^{-1}(0.866) = 30.0^\circ$$

These results are shown in Fig. 2-30c.

EXAMPLE 2.9

Two forces act on the hook shown in Fig. 2–31*a*. Specify the magnitude of $\mathbf{F}_2$ and its coordinate direction angles so that the resultant force $\mathbf{F}_R$ acts along the positive y axis and has a magnitude of 800 N.

SOLUTION

To solve this problem, the resultant force $\mathbf{F}_R$ and its two components, $\mathbf{F}_1$ and $\mathbf{F}_2$, will each be expressed in Cartesian vector form. Then, as shown in Fig. 2–31*b*, it is necessary that $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$.

Applying Eq. 2–9,

$$\begin{aligned}\mathbf{F}_1 &= F_1 \cos \alpha_1 \mathbf{i} + F_1 \cos \beta_1 \mathbf{j} + F_1 \cos \gamma_1 \mathbf{k} \\ &= 300 \cos 45^\circ \mathbf{i} + 300 \cos 60^\circ \mathbf{j} + 300 \cos 120^\circ \mathbf{k} \\ &= \{212.1\mathbf{i} + 150\mathbf{j} - 150\mathbf{k}\} \text{ N} \\ \mathbf{F}_2 &= F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}\end{aligned}$$

Since $\mathbf{F}_R$ has a magnitude of 800 N and acts in the $+\mathbf{j}$ direction,

$$\mathbf{F}_R = (800 \text{ N})(+\mathbf{j}) = \{800\mathbf{j}\} \text{ N}$$

We require

$$\begin{aligned}\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 \\ 800\mathbf{j} &= 212.1\mathbf{i} + 150\mathbf{j} - 150\mathbf{k} + F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k} \\ 800\mathbf{j} &= (212.1 + F_{2x})\mathbf{i} + (150 + F_{2y})\mathbf{j} + (-150 + F_{2z})\mathbf{k}\end{aligned}$$

To satisfy this equation the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components of $\mathbf{F}_R$ must be equal to the corresponding $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components of $(\mathbf{F}_1 + \mathbf{F}_2)$. Hence,

$$\begin{aligned}0 &= 212.1 + F_{2x} & F_{2x} &= -212.1 \text{ N} \\ 800 &= 150 + F_{2y} & F_{2y} &= 650 \text{ N} \\ 0 &= -150 + F_{2z} & F_{2z} &= 150 \text{ N}\end{aligned}$$

The magnitude of $\mathbf{F}_2$ is thus

$$\begin{aligned}F_2 &= \sqrt{(-212.1 \text{ N})^2 + (650 \text{ N})^2 + (150 \text{ N})^2} \\ &= 700 \text{ N}\end{aligned}$$

Ans.

We can use Eq. 2–9 to determine $\alpha_2, \beta_2, \gamma_2$.

$$\cos \alpha_2 = \frac{-212.1}{700}; \quad \alpha_2 = 108^\circ$$

Ans.

$$\cos \beta_2 = \frac{650}{700}; \quad \beta_2 = 21.8^\circ$$

Ans.

$$\cos \gamma_2 = \frac{150}{700}; \quad \gamma_2 = 77.6^\circ$$

Ans.

These results are shown in Fig. 2–31*b*.

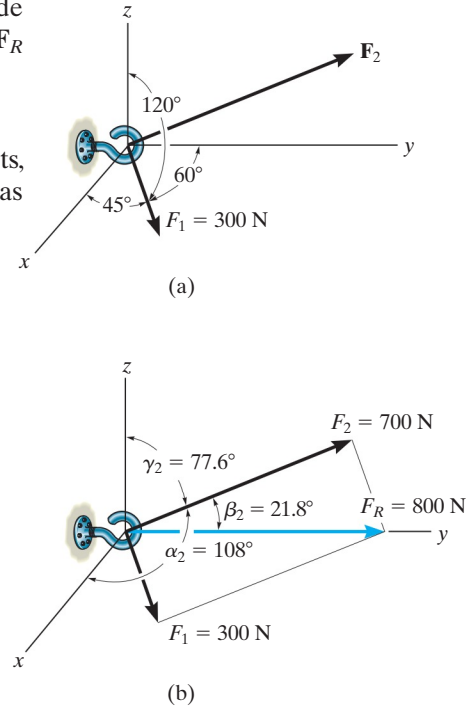


Fig. 2–31

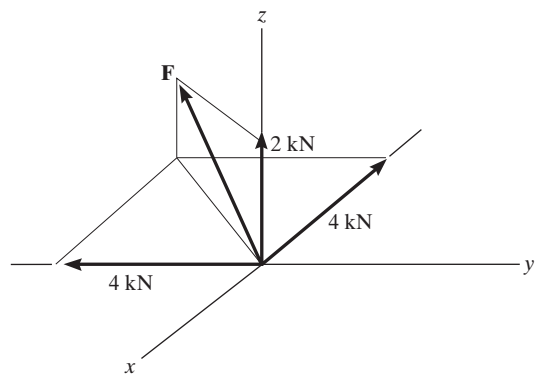
PRELIMINARY PROBLEMS

P2-3. Sketch the following forces on the x, y, z coordinate axes. Show α, β, γ .

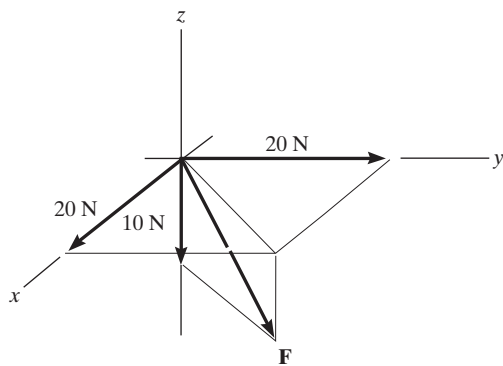
a) $\mathbf{F} = \{50\mathbf{i} + 60\mathbf{j} - 10\mathbf{k}\}$ kN

b) $\mathbf{F} = \{-40\mathbf{i} - 80\mathbf{j} + 60\mathbf{k}\}$ kN

P2-4. In each case, establish $\mathbf{F}$ as a Cartesian vector, and find the magnitude of $\mathbf{F}$ and the direction cosine of β .



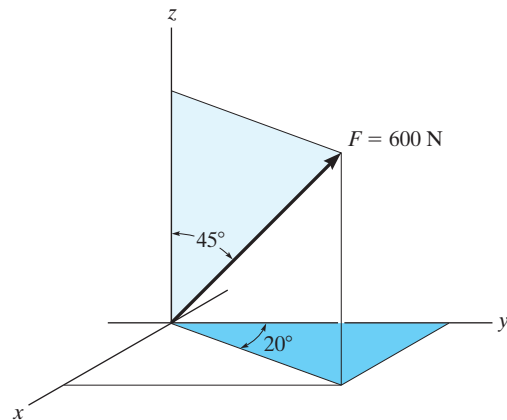
(a)



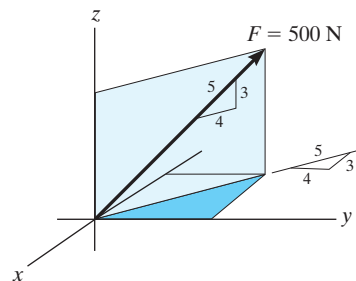
(b)

Prob. P2-4

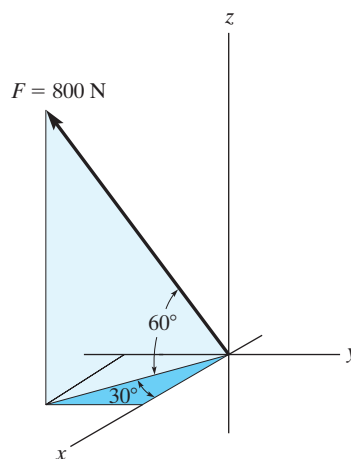
P2-5. Show how to resolve each force into its x, y, z components. Set up the calculation used to find the magnitude of each component.



(a)



(b)

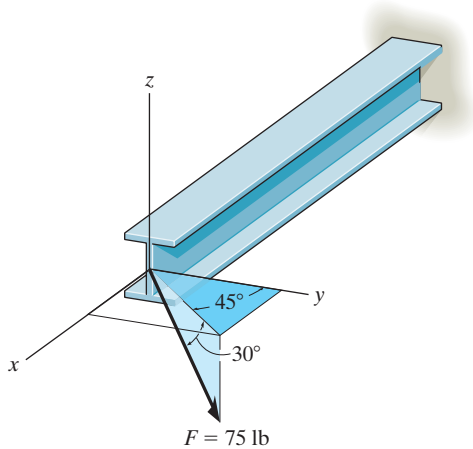


(c)

Prob. P2-5

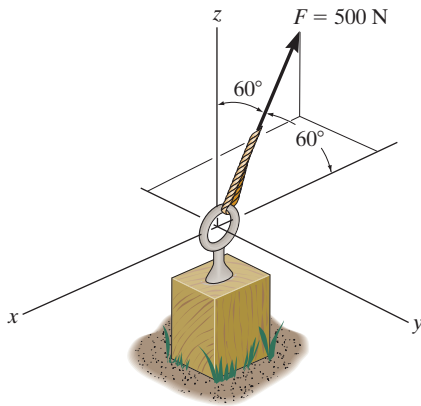
FUNDAMENTAL PROBLEMS

F2-13. Determine the coordinate direction angles of the force.



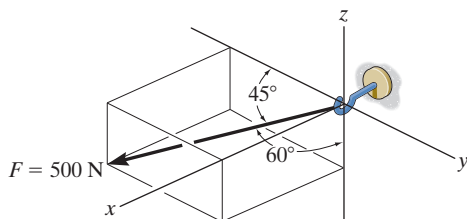
Prob. F2-13

F2-14. Express the force as a Cartesian vector.



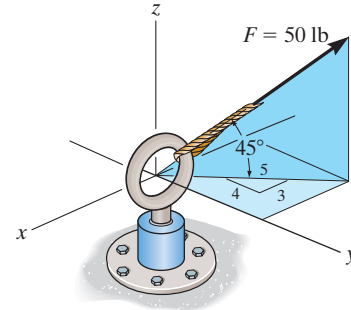
Prob. F2-14

F2-15. Express the force as a Cartesian vector.



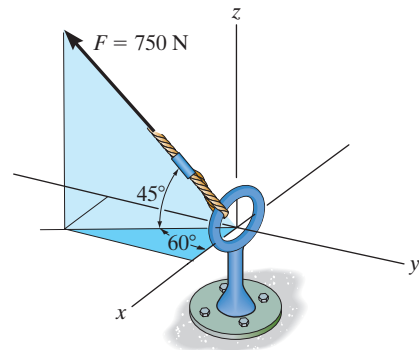
Prob. F2-15

F2-16. Express the force as a Cartesian vector.



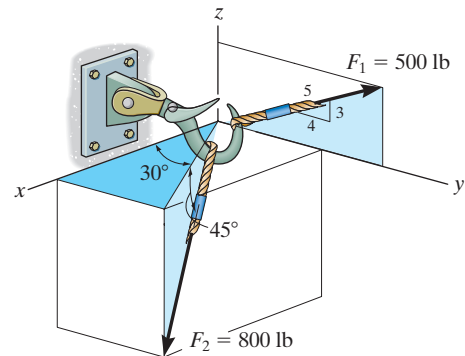
Prob. F2-16

F2-17. Express the force as a Cartesian vector.



Prob. F2-17

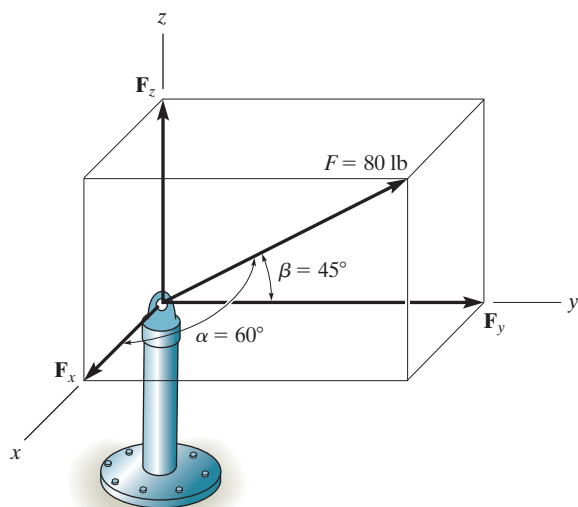
F2-18. Determine the resultant force acting on the hook.



Prob. F2-18

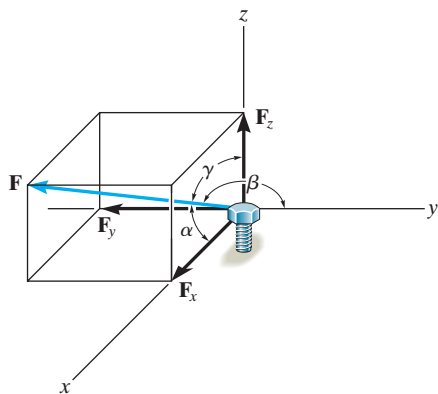
PROBLEMS

***2–60.** The force $\mathbf{F}$ has a magnitude of 80 lb and acts within the octant shown. Determine the magnitudes of the x , y , z components of $\mathbf{F}$.



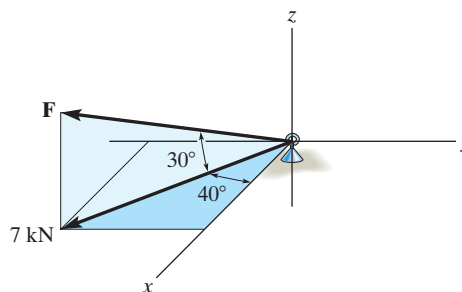
Prob. 2–60

2–61. The bolt is subjected to the force $\mathbf{F}$, which has components acting along the x , y , z axes as shown. If the magnitude of $\mathbf{F}$ is 80 N, and $\alpha = 60^\circ$ and $\gamma = 45^\circ$, determine the magnitudes of its components.



Prob. 2–61

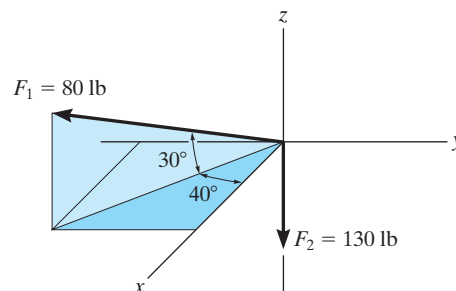
2–62. Determine the magnitude and coordinate direction angles of the force $\mathbf{F}$ acting on the support. The component of $\mathbf{F}$ in the x – y plane is 7 kN.



Prob. 2–62

2–63. Determine the magnitude and coordinate direction angles of the resultant force and sketch this vector on the coordinate system.

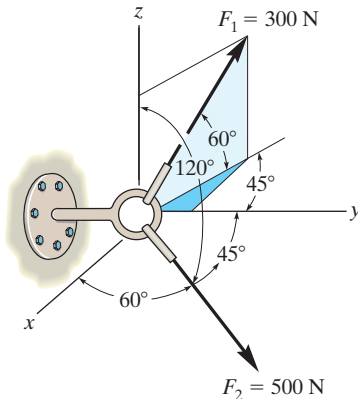
***2–64.** Specify the coordinate direction angles of $\mathbf{F}_1$ and $\mathbf{F}_2$ and express each force as a Cartesian vector.



Probs. 2–63/64

2-65. The screw eye is subjected to the two forces shown. Express each force in Cartesian vector form and then determine the resultant force. Find the magnitude and coordinate direction angles of the resultant force.

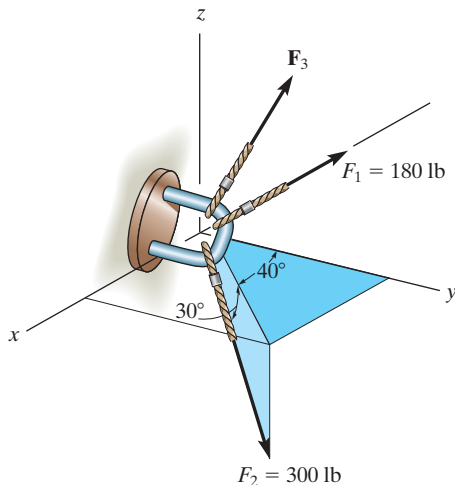
2-66. Determine the coordinate direction angles of $\mathbf{F}_1$.



Probs. 2-65/66

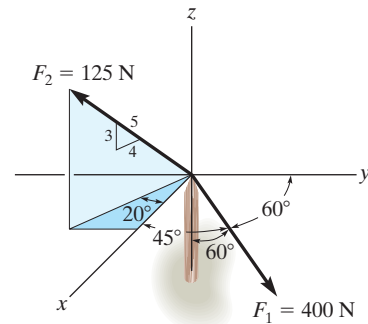
2-67. Determine the magnitude and coordinate direction angles of $\mathbf{F}_3$ so that the resultant of the three forces acts along the positive y axis and has a magnitude of 600 lb.

***2-68.** Determine the magnitude and coordinate direction angles of $\mathbf{F}_3$ so that the resultant of the three forces is zero.



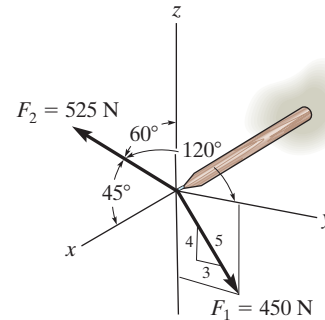
Probs. 2-67/68

2-69. Determine the magnitude and coordinate direction angles of the resultant force, and sketch this vector on the coordinate system.



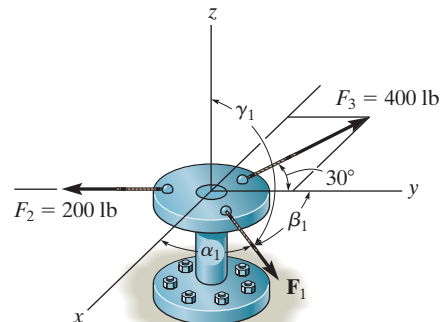
Prob. 2-69

2-70. Determine the magnitude and coordinate direction angles of the resultant force, and sketch this vector on the coordinate system.



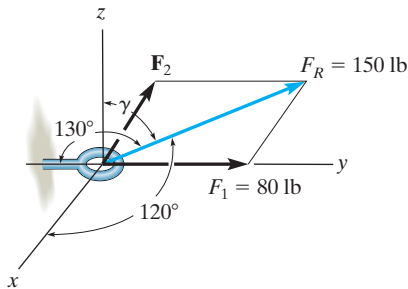
Prob. 2-70

2-71. Specify the magnitude and coordinate direction angles α_1 , β_1 , γ_1 of $\mathbf{F}_1$ so that the resultant of the three forces acting on the bracket is $\mathbf{F}_R = \{-350\mathbf{k}\}$ lb. Note that $\mathbf{F}_3$ lies in the x - y plane.



Prob. 2-71

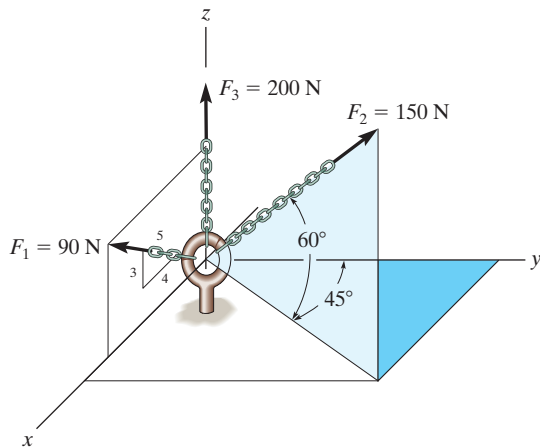
***2-72.** Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the screw eye. If the resultant force $\mathbf{F}_R$ has a magnitude of 150 lb and the coordinate direction angles shown, determine the magnitude of $\mathbf{F}_2$ and its coordinate direction angles.



Prob. 2-72

2-73. Express each force in Cartesian vector form.

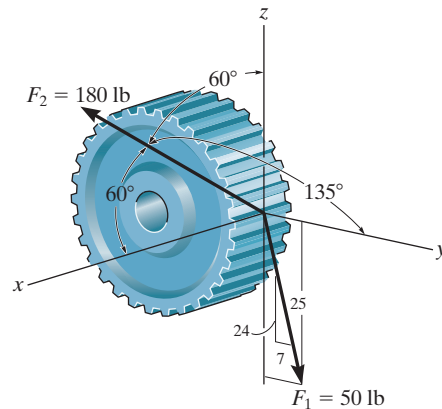
2-74. Determine the magnitude and coordinate direction angles of the resultant force, and sketch this vector on the coordinate system.



Probs. 2-73/74

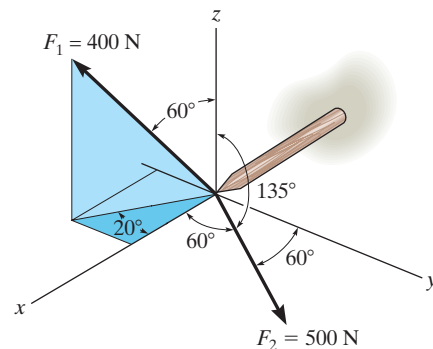
2-75. The spur gear is subjected to the two forces caused by contact with other gears. Express each force as a Cartesian vector.

***2-76.** The spur gear is subjected to the two forces caused by contact with other gears. Determine the resultant of the two forces and express the result as a Cartesian vector.



Probs. 2-75/76

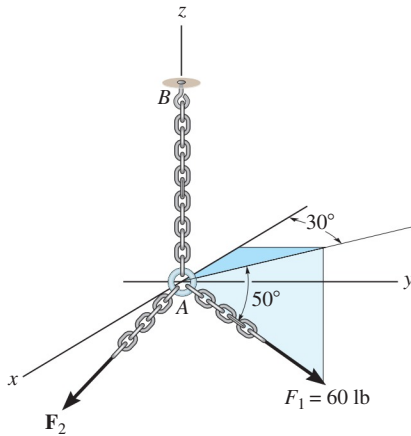
2-77. Determine the magnitude and coordinate direction angles of the resultant force, and sketch this vector on the coordinate system.



Prob. 2-77

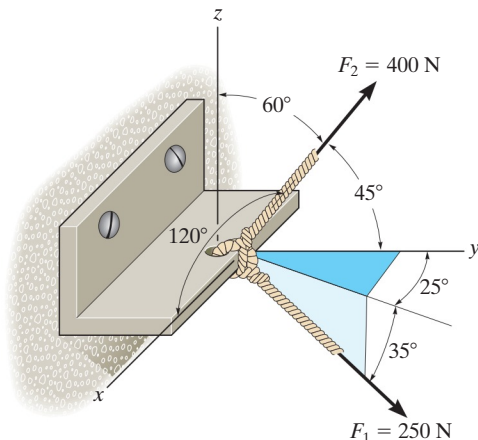
2-78. The two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at A have a resultant force of $\mathbf{F}_R = \{-100\mathbf{k}\}$ lb. Determine the magnitude and coordinate direction angles of $\mathbf{F}_2$.

2-79. Determine the coordinate direction angles of the force $\mathbf{F}_1$ and indicate them on the figure.



Probs. 2-78/79

***2-80.** The bracket is subjected to the two forces shown. Express each force in Cartesian vector form and then determine the resultant force $\mathbf{F}_R$. Find the magnitude and coordinate direction angles of the resultant force.

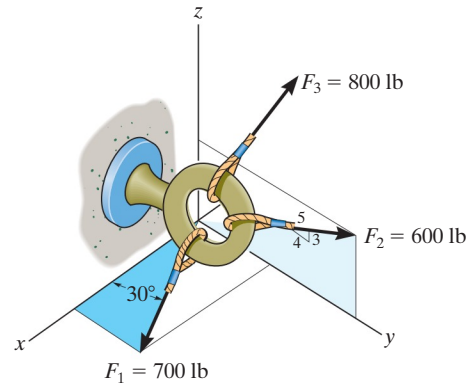


Prob. 2-80

2-81. If the coordinate direction angles for $\mathbf{F}_3$ are $\alpha_3 = 120^\circ$, $\beta_3 = 60^\circ$ and $\gamma_3 = 45^\circ$, determine the magnitude and coordinate direction angles of the resultant force acting on the eyebolt.

2-82. If the coordinate direction angles for $\mathbf{F}_3$ are $\alpha_3 = 120^\circ$, $\beta_3 = 45^\circ$, and $\gamma_3 = 60^\circ$, determine the magnitude and coordinate direction angles of the resultant force acting on the eyebolt.

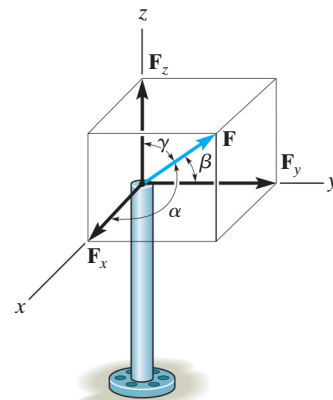
2-83. If the direction of the resultant force acting on the eyebolt is defined by the unit vector $\mathbf{u}_{F_R} = \cos 30^\circ \mathbf{j} + \sin 30^\circ \mathbf{k}$, determine the coordinate direction angles of $\mathbf{F}_3$ and the magnitude of $\mathbf{F}_R$.



Probs. 2-81/82/83

***2-84.** The pole is subjected to the force $\mathbf{F}$, which has components acting along the x , y , z axes as shown. If the magnitude of $\mathbf{F}$ is 3 kN, $\beta = 30^\circ$, and $\gamma = 75^\circ$, determine the magnitudes of its three components.

2-85. The pole is subjected to the force $\mathbf{F}$ which has components $F_x = 1.5$ kN and $F_z = 1.25$ kN. If $\beta = 75^\circ$, determine the magnitudes of $\mathbf{F}$ and $\mathbf{F}_y$.



Probs. 2-84/85