

of an inward normal to the surface of the page. If the velocity of water does not change as we go up- or downstream and also shows no variation as we go across the river (or even if it decreases in the same fashion toward either bank), then this component is the only component present at the center of the stream, and the curl of the water velocity has a direction into the page.

In Figure 7.14*b*, the streamlines of the magnetic field intensity about an infinitely long filamentary conductor are shown. The curl meter placed in this field of curved lines shows that a larger number of blades have a clockwise force exerted on them but that this force is in general smaller than the counterclockwise force exerted on the smaller number of blades closer to the wire. It seems possible that if the curvature of the streamlines is correct and also if the variation of the field strength is just right, the net torque on the paddle wheel may be zero. Actually, the paddle wheel does not rotate in this case, for since $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$, we may substitute into (25) obtaining

$$\text{curl } \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z = 0$$

EXAMPLE 7.2

As an example of the evaluation of $\text{curl } \mathbf{H}$ from the definition and of the evaluation of another line integral, suppose that $\mathbf{H} = 0.2z^2\mathbf{a}_x$ for $z > 0$, and $\mathbf{H} = 0$ elsewhere, as shown in Figure 7.15. Calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ about a square path with side d , centered at $(0, 0, z_1)$ in the $y = 0$ plane where $z_1 > d/2$.

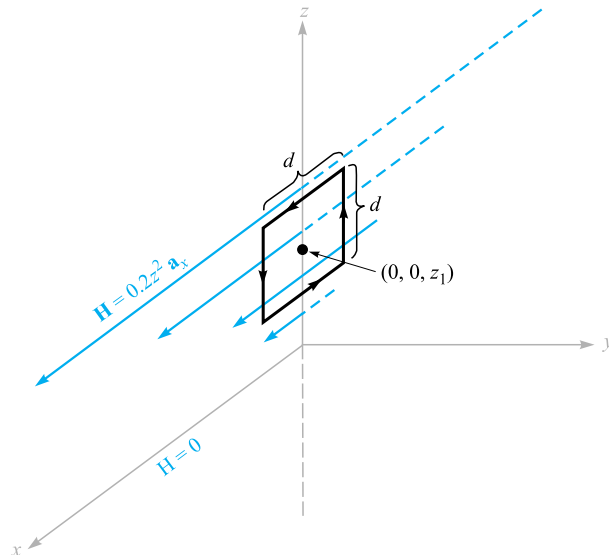


Figure 7.15 A square path of side d with its center on the z axis at $z = z_1$ is used to evaluate $\oint \mathbf{H} \cdot d\mathbf{L}$ and find $\text{curl } \mathbf{H}$.

Solution. We evaluate the line integral of $\mathbf{H}$ along the four segments, beginning at the top:

$$\begin{aligned}\oint \mathbf{H} \cdot d\mathbf{L} &= 0.2 \left(z_1 + \frac{1}{2}d\right)^2 d + 0 - 0.2 \left(z_1 - \frac{1}{2}d\right)^2 d + 0 \\ &= 0.4z_1 d^2\end{aligned}$$

In the limit as the area approaches zero, we find

$$(\nabla \times \mathbf{H})_y = \lim_{d \rightarrow 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{d^2} = \lim_{d \rightarrow 0} \frac{0.4z_1 d^2}{d^2} = 0.4z_1$$

The other components are zero, so $\nabla \times \mathbf{H} = 0.4z_1 \mathbf{a}_y$.

To evaluate the curl without trying to illustrate the definition or the evaluation of a line integral, we simply take the partial derivative indicated by (23):

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0.2z^2 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z}(0.2z^2)\mathbf{a}_y = 0.4z\mathbf{a}_y$$

which checks with the preceding result when $z = z_1$.

Returning now to complete our original examination of the application of Ampère's circuital law to a differential-sized path, we may combine (18)–(20), (22), and (24),

$$\begin{aligned}\text{curl } \mathbf{H} = \nabla \times \mathbf{H} &= \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y \\ &\quad + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J}\end{aligned}\tag{27}$$

and write the *point form of Ampère's circuital law*,

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}\tag{28}$$

This is the second of Maxwell's four equations as they apply to non-time-varying conditions. We may also write the third of these equations at this time; it is the point form of $\oint \mathbf{E} \cdot d\mathbf{L} = 0$, or

$$\boxed{\nabla \times \mathbf{E} = 0}\tag{29}$$

The fourth equation appears in Section 7.5.

D7.4. (a) Evaluate the closed line integral of $\mathbf{H}$ about the rectangular path $P_1(2, 3, 4)$ to $P_2(4, 3, 4)$ to $P_3(4, 3, 1)$ to $P_4(2, 3, 1)$ to P_1 , given $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$ A/m. (b) Determine the quotient of the closed line integral and the area enclosed by the path as an approximation to $(\nabla \times \mathbf{H})_y$. (c) Determine $(\nabla \times \mathbf{H})_y$ at the center of the area.

Ans. 354 A; 59 A/m²; 57 A/m²

D7.5. Calculate the value of the vector current density: (a) in rectangular coordinates at $P_A(2, 3, 4)$ if $\mathbf{H} = x^2z\mathbf{a}_y - y^2x\mathbf{a}_z$; (b) in cylindrical coordinates at $P_B(1.5, 90^\circ, 0.5)$ if $\mathbf{H} = \frac{2}{\rho}(\cos 0.2\phi)\mathbf{a}_\rho$; (c) in spherical coordinates at $P_C(2, 30^\circ, 20^\circ)$ if $\mathbf{H} = \frac{1}{\sin \theta}\mathbf{a}_\theta$.

Ans. $-16\mathbf{a}_x + 9\mathbf{a}_y + 16\mathbf{a}_z$ A/m²; $0.055\mathbf{a}_z$ A/m²; $\mathbf{a}_\phi$ A/m²

7.4 STOKES' THEOREM

Although Section 7.3 was devoted primarily to a discussion of the curl operation, the contribution to the subject of magnetic fields should not be overlooked. From Ampère's circuital law we derived one of Maxwell's equations, $\nabla \times \mathbf{H} = \mathbf{J}$. This latter equation should be considered the point form of Ampère's circuital law and applies on a "per-unit-area" basis. In this section we shall again devote a major share of the material to the mathematical theorem known as Stokes' theorem, but in the process we will show that we may obtain Ampère's circuital law from $\nabla \times \mathbf{H} = \mathbf{J}$. In other words, we are then prepared to obtain the integral form from the point form or to obtain the point form from the integral form.

Consider the surface S of Figure 7.16, which is broken up into incremental surfaces of area ΔS . If we apply the definition of the curl to one of these incremental surfaces, then

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H})_N$$

where the N subscript again indicates the right-hand normal to the surface. The subscript on $d\mathbf{L}_{\Delta S}$ indicates that the closed path is the perimeter of an incremental area ΔS . This result may also be written

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N$$

or

$$\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N \Delta S = (\nabla \times \mathbf{H}) \cdot \Delta \mathbf{S}$$

where $\mathbf{a}_N$ is a unit vector in the direction of the right-hand normal to ΔS .

Now let us determine this circulation for every ΔS comprising S and sum the results. As we evaluate the closed line integral for each ΔS , some cancellation will occur

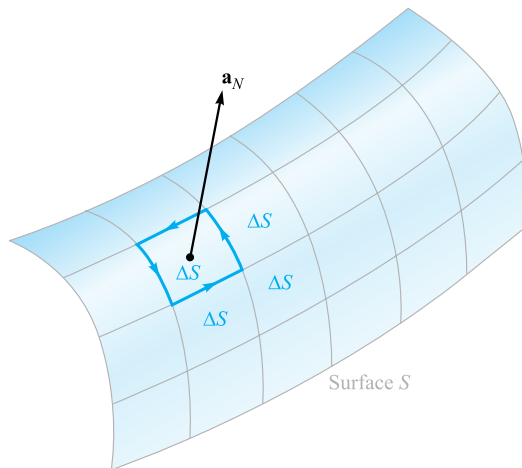


Figure 7.16 The sum of the closed line integrals about the perimeter of every ΔS is the same as the closed line integral about the perimeter of S because of cancellation on every interior path.

because every *interior* wall is covered once in each direction. The only boundaries on which cancellation cannot occur form the outside boundary, the path enclosing S . Therefore we have

$$\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} \quad (30)$$

where $d\mathbf{L}$ is taken only on the perimeter of S .

Equation (30) is an identity, holding for any vector field, and is known as *Stokes' theorem*.

EXAMPLE 7.3

A numerical example may help to illustrate the geometry involved in Stokes' theorem. Consider the portion of a sphere shown in Figure 7.17. The surface is specified by $r = 4$, $0 \leq \theta \leq 0.1\pi$, $0 \leq \phi \leq 0.3\pi$, and the closed path forming its perimeter is composed of three circular arcs. We are given the field $\mathbf{H} = 6r \sin \phi \mathbf{a}_r + 18r \sin \theta \cos \phi \mathbf{a}_\phi$ and are asked to evaluate each side of Stokes' theorem.

Solution. The first path segment is described in spherical coordinates by $r = 4$, $0 \leq \theta \leq 0.1\pi$, $\phi = 0$; the second one by $r = 4$, $\theta = 0.1\pi$, $0 \leq \phi \leq 0.3\pi$; and the third by $r = 4$, $0 \leq \theta \leq 0.1\pi$, $\phi = 0.3\pi$. The differential path element $d\mathbf{L}$ is the vector sum of the three differential lengths of the spherical coordinate system first discussed in Section 1.9,

$$d\mathbf{L} = dr \mathbf{a}_r + r d\theta \mathbf{a}_\theta + r \sin \theta d\phi \mathbf{a}_\phi$$

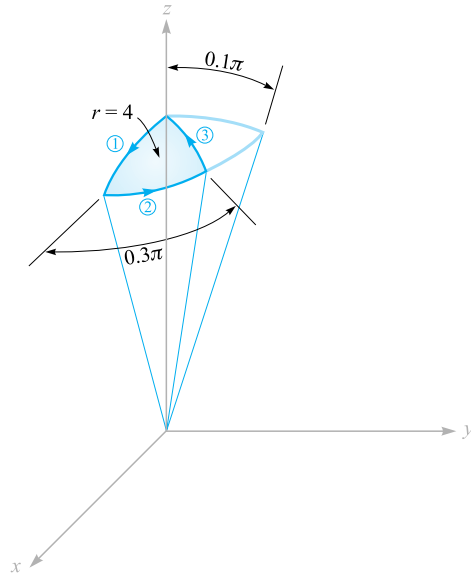


Figure 7.17 A portion of a spherical cap is used as a surface and a closed path to illustrate Stokes' theorem.

The first term is zero on all three segments of the path since $r = 4$ and $dr = 0$, the second is zero on segment 2 as θ is constant, and the third term is zero on both segments 1 and 3. Thus,

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_1 H_\theta r d\theta + \int_2 H_\phi r \sin \theta d\phi + \int_3 H_\theta r d\theta$$

Because $H_\theta = 0$, we have only the second integral to evaluate,

$$\begin{aligned} \oint \mathbf{H} \cdot d\mathbf{L} &= \int_0^{0.3\pi} [18(4) \sin 0.1\pi \cos \phi] 4 \sin 0.1\pi d\phi \\ &= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A} \end{aligned}$$

We next attack the surface integral. First, we use (26) to find

$$\nabla \times \mathbf{H} = \frac{1}{r \sin \theta} (36r \sin \theta \cos \theta \cos \phi) \mathbf{a}_r + \frac{1}{r} \left(\frac{1}{\sin \theta} 6r \cos \phi - 36r \sin \theta \cos \phi \right) \mathbf{a}_\theta$$

Because $d\mathbf{S} = r^2 \sin \theta d\theta d\phi \mathbf{a}_r$, the integral is

$$\begin{aligned} \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} &= \int_0^{0.3\pi} \int_0^{0.1\pi} (36 \cos \theta \cos \phi) 16 \sin \theta d\theta d\phi \\ &= \int_0^{0.3\pi} 576 \left(\frac{1}{2} \sin^2 \theta \right) \Big|_0^{0.1\pi} \cos \phi d\phi \\ &= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A} \end{aligned}$$

Thus, the results check Stokes' theorem, and we note in passing that a current of 22.2 A is flowing upward through this section of a spherical cap.

Next, let us see how easy it is to obtain Ampère's circuital law from $\nabla \times \mathbf{H} = \mathbf{J}$. We merely have to dot each side by $d\mathbf{S}$, integrate each side over the same (open) surface S , and apply Stokes' theorem:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{L}$$

The integral of the current density over the surface S is the total current I passing through the surface, and therefore

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

This short derivation shows clearly that the current I , described as being "enclosed by the closed path," is also the current passing through any of the infinite number of surfaces that have the closed path as a perimeter.

Stokes' theorem relates a surface integral to a closed line integral. It should be recalled that the divergence theorem relates a volume integral to a closed surface integral. Both theorems find their greatest use in general vector proofs. As an example, let us find another expression for $\nabla \cdot \nabla \times \mathbf{A}$, where $\mathbf{A}$ represents any vector field. The result must be a scalar (why?), and we may let this scalar be T , or

$$\nabla \cdot \nabla \times \mathbf{A} = T$$

Multiplying by $d\nu$ and integrating throughout any volume ν ,

$$\int_{\text{vol}} (\nabla \cdot \nabla \times \mathbf{A}) d\nu = \int_{\text{vol}} T d\nu$$

we first apply the divergence theorem to the left side, obtaining

$$\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{\text{vol}} T d\nu$$

The left side is the surface integral of the curl of $\mathbf{A}$ over the *closed* surface surrounding the volume ν . Stokes' theorem relates the surface integral of the curl of $\mathbf{A}$ over the *open* surface enclosed by a given closed path. If we think of the path as the opening of a laundry bag and the open surface as the surface of the bag itself, we see that as we gradually approach a closed surface by pulling on the drawstrings, the closed path becomes smaller and smaller and finally disappears as the surface becomes closed. Hence, the application of Stokes' theorem to a *closed* surface produces a zero result, and we have

$$\int_{\text{vol}} T d\nu = 0$$

Because this is true for any volume, it is true for the differential volume $d\nu$,

$$T d\nu = 0$$

and therefore

$$T = 0$$

or

$$\nabla \cdot \nabla \times \mathbf{A} \equiv 0 \quad (31)$$

Equation (31) is a useful identity of vector calculus.⁶ Of course, it may also be proven easily by direct expansion in rectangular coordinates.

Let us apply the identity to the non-time-varying magnetic field for which

$$\nabla \times \mathbf{H} = \mathbf{J}$$

This shows quickly that

$$\nabla \cdot \mathbf{J} = 0$$

which is the same result we obtained earlier in the chapter by using the continuity equation.

Before introducing several new magnetic field quantities in the following section, we may review our accomplishments at this point. We initially accepted the Biot-Savart law as an experimental result,

$$\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

and tentatively accepted Ampère's circuital law, subject to later proof,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

From Ampère's circuital law the definition of curl led to the point form of this same law,

$$\nabla \times \mathbf{H} = \mathbf{J}$$

We now see that Stokes' theorem enables us to obtain the integral form of Ampère's circuital law from the point form.

D7.6. Evaluate both sides of Stokes' theorem for the field $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m and the rectangular path around the region, $2 \leq x \leq 5$, $-1 \leq y \leq 1$, $z = 0$. Let the positive direction of $d\mathbf{S}$ be $\mathbf{a}_z$.

Ans. -126 A; -126 A

⁶ This and other vector identities are tabulated in Appendix A.3.

7.5 MAGNETIC FLUX AND MAGNETIC FLUX DENSITY

In free space, let us define the *magnetic flux density* $\mathbf{B}$ as

$$\mathbf{B} = \mu_0 \mathbf{H} \quad (\text{free space only}) \quad (32)$$

where $\mathbf{B}$ is measured in webers per square meter (Wb/m^2) or in a newer unit adopted in the International System of Units, tesla (T). An older unit that is often used for magnetic flux density is the gauss (G), where 1 T or 1 Wb/m^2 is the same as 10,000 G. The constant μ_0 is not dimensionless and has the *defined value* for free space, in henrys per meter (H/m), of

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m} \quad (33)$$

The name given to μ_0 is the *permeability* of free space.

We should note that since $\mathbf{H}$ is measured in amperes per meter, the weber is dimensionally equal to the product of henrys and amperes. Considering the henry as a new unit, the weber is merely a convenient abbreviation for the product of henrys and amperes. When time-varying fields are introduced, it will be shown that a weber is also equivalent to the product of volts and seconds.

The magnetic-flux-density vector $\mathbf{B}$, as the name weber per square meter implies, is a member of the flux-density family of vector fields. One of the possible analogies between electric and magnetic fields⁷ compares the laws of Biot-Savart and Coulomb, thus establishing an analogy between $\mathbf{H}$ and $\mathbf{E}$. The relations $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \epsilon_0 \mathbf{E}$ then lead to an analogy between $\mathbf{B}$ and $\mathbf{D}$. If $\mathbf{B}$ is measured in teslas or webers per square meter, then magnetic flux should be measured in webers. Let us represent magnetic flux by Φ and define Φ as the flux passing through any designated area,

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \text{ Wb} \quad (34)$$

Our analogy should now remind us of the electric flux Ψ , measured in coulombs, and of Gauss's law, which states that the total flux passing through any closed surface is equal to the charge enclosed,

$$\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

The charge Q is the source of the lines of electric flux and these lines begin and terminate on positive and negative charges, respectively.

⁷ An alternate analogy is presented in Section 9.2.

No such source has ever been discovered for the lines of magnetic flux. In the example of the infinitely long straight filament carrying a direct current I , the $\mathbf{H}$ field formed concentric circles about the filament. Because $\mathbf{B} = \mu_0 \mathbf{H}$, the $\mathbf{B}$ field is of the same form. The magnetic flux lines are closed and do not terminate on a “magnetic charge.” For this reason Gauss’s law for the magnetic field is

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad (35)$$

and application of the divergence theorem shows us that

$$\nabla \cdot \mathbf{B} = 0 \quad (36)$$

Equation (36) is the last of Maxwell’s four equations as they apply to static electric fields and steady magnetic fields. Collecting these equations, we then have for static electric fields and steady magnetic fields

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_v \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \quad (37)$$

To these equations we may add the two expressions relating $\mathbf{D}$ to $\mathbf{E}$ and $\mathbf{B}$ to $\mathbf{H}$ in free space,

$$\mathbf{D} = \epsilon_0 \mathbf{E} \quad (38)$$

$$\mathbf{B} = \mu_0 \mathbf{H} \quad (39)$$

We have also found it helpful to define an electrostatic potential,

$$\mathbf{E} = -\nabla V \quad (40)$$

and we will discuss a potential for the steady magnetic field in the following section. In addition, we extended our coverage of electric fields to include conducting materials and dielectrics, and we introduced the polarization $\mathbf{P}$. A similar treatment will be applied to magnetic fields in the next chapter.

Returning to (37), it may be noted that these four equations specify the divergence and curl of an electric and a magnetic field. The corresponding set of four integral

equations that apply to static electric fields and steady magnetic fields is

$$\begin{aligned}
 \oint_S \mathbf{D} \cdot d\mathbf{S} &= Q = \int_{\text{vol}} \rho_v dv \\
 \oint \mathbf{E} \cdot d\mathbf{L} &= 0 \\
 \oint \mathbf{H} \cdot d\mathbf{L} &= I = \int_S \mathbf{J} \cdot d\mathbf{S} \\
 \oint_S \mathbf{B} \cdot d\mathbf{S} &= 0
 \end{aligned}
 \tag{41}$$

Our study of electric and magnetic fields would have been much simpler if we could have begun with either set of equations, (37) or (41). With a good knowledge of vector analysis, such as we should now have, either set may be readily obtained from the other by applying the divergence theorem or Stokes' theorem. The various experimental laws can be obtained easily from these equations.

As an example of the use of flux and flux density in magnetic fields, let us find the flux between the conductors of the coaxial line of Figure 7.8a. The magnetic field intensity was found to be

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

and therefore

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho} \mathbf{a}_\phi$$

The magnetic flux contained between the conductors in a length d is the flux crossing any radial plane extending from $\rho = a$ to $\rho = b$ and from, say, $z = 0$ to $z = d$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho} \mathbf{a}_\phi \cdot d\rho dz \mathbf{a}_\phi$$

or

$$\Phi = \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a} \tag{42}$$

This expression will be used later to obtain the inductance of the coaxial transmission line.

D7.7. A solid conductor of circular cross section is made of a homogeneous nonmagnetic material. If the radius $a = 1$ mm, the conductor axis lies on the z axis, and the total current in the $\mathbf{a}_z$ direction is 20 A, find: (a) H_ϕ at $\rho = 0.5$ mm; (b) B_ϕ at $\rho = 0.8$ mm; (c) the total magnetic flux per unit length inside the conductor; (d) the total flux for $\rho < 0.5$ mm; (e) the total magnetic flux outside the conductor.

Ans. 1592 A/m; 3.2 mT; $2 \mu\text{Wb/m}$; $0.5 \mu\text{Wb}$; ∞

7.6 THE SCALAR AND VECTOR MAGNETIC POTENTIALS

The solution of electrostatic field problems is greatly simplified by the use of the scalar electrostatic potential V . Although this potential possesses a very real physical significance for us, it is mathematically no more than a stepping-stone which allows us to solve a problem by several smaller steps. Given a charge configuration, we may first find the potential and then from it the electric field intensity.

We should question whether or not such assistance is available in magnetic fields. Can we define a potential function which may be found from the current distribution and from which the magnetic fields may be easily determined? Can a scalar magnetic potential be defined, similar to the scalar electrostatic potential? We will show in the next few pages that the answer to the first question is yes, but the second must be answered “sometimes.” Let us attack the second question first by assuming the existence of a scalar magnetic potential, which we designate V_m , whose negative gradient gives the magnetic field intensity

$$\mathbf{H} = -\nabla V_m$$

The selection of the negative gradient provides a closer analogy to the electric potential and to problems which we have already solved.

This definition must not conflict with our previous results for the magnetic field, and therefore

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m)$$

However, the curl of the gradient of any scalar is identically zero, a vector identity the proof of which is left for a leisure moment. Therefore, we see that if $\mathbf{H}$ is to be defined as the gradient of a scalar magnetic potential, then current density must be zero throughout the region in which the scalar magnetic potential is so defined. We then have

$$\mathbf{H} = -\nabla V_m \quad (\mathbf{J} = 0) \quad (43)$$

Because many magnetic problems involve geometries in which the current-carrying conductors occupy a relatively small fraction of the total region of interest, it is evident that a scalar magnetic potential can be useful. The scalar magnetic potential is also applicable in the case of permanent magnets. The dimensions of V_m are obviously amperes.

This scalar potential also satisfies Laplace’s equation. In free space,

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = 0$$

and hence

$$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

or

$$\nabla^2 V_m = 0 \quad (\mathbf{J} = 0) \quad (44)$$

We will see later that V_m continues to satisfy Laplace's equation in homogeneous magnetic materials; it is not defined in any region in which current density is present.

Although we shall consider the scalar magnetic potential to a much greater extent in Chapter 8, when we introduce magnetic materials and discuss the magnetic circuit, one difference between V and V_m should be pointed out now: V_m is not a single-valued function of position. The electric potential V is single-valued; once a zero reference is assigned, there is only one value of V associated with each point in space. Such is not the case with V_m . Consider the cross section of the coaxial line shown in Figure 7.18. In the region $a < \rho < b$, $\mathbf{J} = 0$, and we may establish a scalar magnetic potential. The value of $\mathbf{H}$ is

$$\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi$$

where I is the total current flowing in the $\mathbf{a}_z$ direction in the inner conductor. We find V_m by integrating the appropriate component of the gradient. Applying (43),

$$\frac{I}{2\pi\rho} = -\nabla V_m \Big|_\phi = -\frac{1}{\rho} \frac{\partial V_m}{\partial \phi}$$

or

$$\frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi}$$

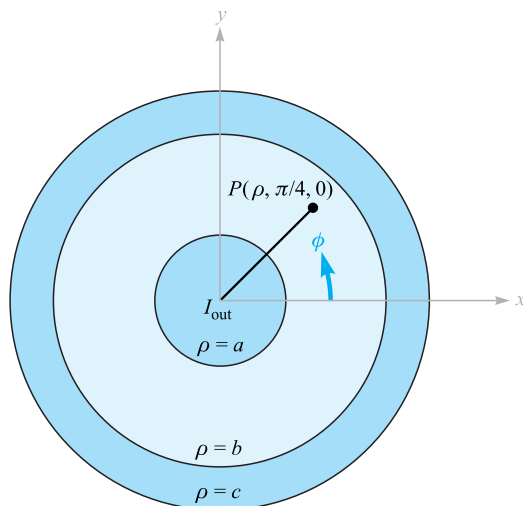


Figure 7.18 The scalar magnetic potential V_m is a multivalued function of ϕ in the region $a < \rho < b$. The electrostatic potential is always single valued.

Thus,

$$V_m = -\frac{I}{2\pi}\phi$$

where the constant of integration has been set equal to zero. What value of potential do we associate with point P , where $\phi = \pi/4$? If we let V_m be zero at $\phi = 0$ and proceed counterclockwise around the circle, the magnetic potential goes negative linearly. When we have made one circuit, the potential is $-I$, but that was the point at which we said the potential was zero a moment ago. At P , then, $\phi = \pi/4, 9\pi/4, 17\pi/4, \dots$, or $-7\pi/4, -15\pi/4, -23\pi/4, \dots$, or

$$V_{mP} = \frac{I}{2\pi} \left(2n - \frac{1}{4}\right) \pi \quad (n = 0, \pm 1, \pm 2, \dots)$$

or

$$V_{mP} = I \left(n - \frac{1}{8}\right) \quad (n = 0, \pm 1, \pm 2, \dots)$$

The reason for this multivaluedness may be shown by a comparison with the electrostatic case. There, we know that

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 \\ \oint \mathbf{E} \cdot d\mathbf{L} &= 0 \end{aligned}$$

and therefore the line integral

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{L}$$

is independent of the path. In the magnetostatic case, however,

$$\nabla \times \mathbf{H} = 0 \quad (\text{wherever } \mathbf{J} = 0)$$

but

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

even if $\mathbf{J}$ is zero along the path of integration. Every time we make another complete lap around the current, the result of the integration increases by I . If no current I is enclosed by the path, then a single-valued potential function may be defined. In general, however,

$$V_{m,ab} = -\int_b^a \mathbf{H} \cdot d\mathbf{L} \quad (\text{specified path}) \quad (45)$$

where a specific path or type of path must be selected. We should remember that the electrostatic potential V is a conservative field; the magnetic scalar potential V_m is not a conservative field. In our coaxial problem, let us erect a barrier⁸ at $\phi = \pi$; we

⁸ This corresponds to the more precise mathematical term “branch cut.”

agree not to select a path that crosses this plane. Therefore, we cannot encircle I , and a single-valued potential is possible. The result is seen to be

$$V_m = -\frac{I}{2\pi}\phi \quad (-\pi < \phi < \pi)$$

and

$$V_{mP} = -\frac{I}{8} \quad \left(\phi = \frac{\pi}{4}\right)$$

The scalar magnetic potential is evidently the quantity whose equipotential surfaces will form curvilinear squares with the streamlines of $\mathbf{H}$ in Figure 7.4. This is one more facet of the analogy between electric and magnetic fields about which we will have more to say in the next chapter.

Let us temporarily leave the scalar magnetic potential now and investigate a vector magnetic potential. This vector field is one which is extremely useful in studying radiation from antennas (as we will find in Chapter 14) as well as radiation leakage from transmission lines, waveguides, and microwave ovens. The vector magnetic potential may be used in regions where the current density is zero or nonzero, and we shall also be able to extend it to the time-varying case later.

Our choice of a vector magnetic potential is indicated by noting that

$$\nabla \cdot \mathbf{B} = 0$$

Next, a vector identity that we proved in Section 7.4 shows that the divergence of the curl of any vector field is zero. Therefore, we select

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (46)$$

where $\mathbf{A}$ signifies a *vector magnetic potential*, and we automatically satisfy the condition that the magnetic flux density shall have zero divergence. The $\mathbf{H}$ field is

$$\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A}$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A}$$

The curl of the curl of a vector field is not zero and is given by a fairly complicated expression,⁹ which we need not know now in general form. In specific cases for which the form of $\mathbf{A}$ is known, the curl operation may be applied twice to determine the current density.

⁹ $\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$. In rectangular coordinates, it may be shown that $\nabla^2 \mathbf{A} \equiv \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$. In other coordinate systems, $\nabla^2 \mathbf{A}$ may be found by evaluating the second-order partial derivatives in $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$.

Equation (46) serves as a useful definition of the *vector magnetic potential* $\mathbf{A}$. Because the curl operation implies differentiation with respect to a length, the units of $\mathbf{A}$ are webers per meter.

As yet we have seen only that the definition for $\mathbf{A}$ does not conflict with any previous results. It still remains to show that this particular definition can help us to determine magnetic fields more easily. We certainly cannot identify $\mathbf{A}$ with any easily measured quantity or history-making experiment.

We will show in Section 7.7 that, given the Biot-Savart law, the definition of $\mathbf{B}$, and the definition of $\mathbf{A}$, $\mathbf{A}$ may be determined from the differential current elements by

$$\mathbf{A} = \oint \frac{\mu_0 I d\mathbf{L}}{4\pi R} \quad (47)$$

The significance of the terms in (47) is the same as in the Biot-Savart law; a direct current I flows along a filamentary conductor of which any differential length $d\mathbf{L}$ is distant R from the point at which $\mathbf{A}$ is to be found. Because we have defined $\mathbf{A}$ only through specification of its curl, it is possible to add the gradient of any scalar field to (47) without changing $\mathbf{B}$ or $\mathbf{H}$, for the curl of the gradient is identically zero. In steady magnetic fields, it is customary to set this possible added term equal to zero.

The fact that $\mathbf{A}$ is a vector magnetic *potential* is more apparent when (47) is compared with the similar expression for the electrostatic potential,

$$V = \int \frac{\rho_L dL}{4\pi\epsilon_0 R}$$

Each expression is the integral along a line source, in one case line charge and in the other case line current; each integrand is inversely proportional to the distance from the source to the point of interest; and each involves a characteristic of the medium (here free space), the permeability or the permittivity.

Equation (47) may be written in differential form,

$$d\mathbf{A} = \frac{\mu_0 I d\mathbf{L}}{4\pi R} \quad (48)$$

if we again agree not to attribute any physical significance to any magnetic fields we obtain from (48) until the *entire closed path in which the current flows is considered*.

With this reservation, let us go right ahead and consider the vector magnetic potential field about a differential filament. We locate the filament at the origin in free space, as shown in Figure 7.19, and allow it to extend in the positive z direction so that $d\mathbf{L} = dz \mathbf{a}_z$. We use cylindrical coordinates to find $d\mathbf{A}$ at the point (ρ, ϕ, z) :

$$d\mathbf{A} = \frac{\mu_0 I dz \mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$$

or

$$d\mathbf{A}_z = \frac{\mu_0 I dz}{4\pi\sqrt{\rho^2 + z^2}} \quad dA_\phi = 0 \quad dA_\rho = 0 \quad (49)$$

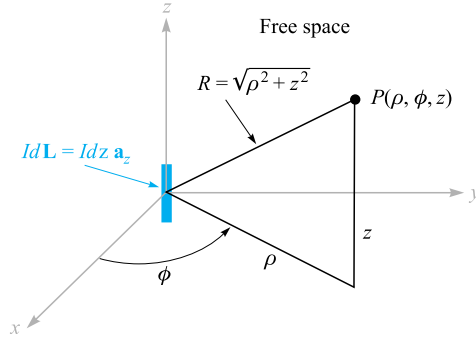


Figure 7.19 The differential current element $I dz \mathbf{a}_z$ at the origin establishes the differential vector magnetic potential field, $d\mathbf{A} = \frac{\mu_0 I dz \mathbf{a}_z}{4\pi \sqrt{\rho^2 + z^2}}$ at $P(\rho, \phi, z)$.

We note that the direction of $d\mathbf{A}$ is the same as that of $I d\mathbf{L}$. Each small section of a current-carrying conductor produces a contribution to the total vector magnetic potential which is in the same direction as the current flow in the conductor. The magnitude of the vector magnetic potential varies inversely with the distance to the current element, being strongest in the neighborhood of the current and gradually falling off to zero at distant points. Skilling¹⁰ describes the vector magnetic potential field as “like the current distribution but fuzzy around the edges, or like a picture of the current out of focus.”

In order to find the magnetic field intensity, we must take the curl of (49) in cylindrical coordinates, leading to

$$d\mathbf{H} = \frac{1}{\mu_0} \nabla \times d\mathbf{A} = \frac{1}{\mu_0} \left(-\frac{\partial dA_z}{\partial \rho} \right) \mathbf{a}_\phi$$

or

$$d\mathbf{H} = \frac{I dz}{4\pi} \frac{\rho}{(\rho^2 + z^2)^{3/2}} \mathbf{a}_\phi$$

which is easily shown to be the same as the value given by the Biot-Savart law.

Expressions for the vector magnetic potential $\mathbf{A}$ can also be obtained for a current source which is distributed. For a current sheet $\mathbf{K}$, the differential current element becomes

$$I d\mathbf{L} = \mathbf{K} dS$$

In the case of current flow throughout a volume with a density $\mathbf{J}$, we have

$$I d\mathbf{L} = \mathbf{J} dv$$

¹⁰ See the References at the end of the chapter.



In each of these two expressions the vector character is given to the current. For the filamentary element it is customary, although not necessary, to use $I d\mathbf{L}$ instead of $\mathbf{I} dL$. Since the magnitude of the filamentary element is constant, we have chosen the form which allows us to remove one quantity from the integral. The alternative expressions for $\mathbf{A}$ are then

$$\mathbf{A} = \int_S \frac{\mu_0 \mathbf{K} dS}{4\pi R} \quad (50)$$

and

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu_0 \mathbf{J} dv}{4\pi R} \quad (51)$$

Equations (47), (50), and (51) express the vector magnetic potential as an integration over all of its sources. From a comparison of the form of these integrals with those which yield the electrostatic potential, it is evident that once again the zero reference for $\mathbf{A}$ is at infinity, for no finite current element can produce any contribution as $R \rightarrow \infty$. We should remember that we very seldom used the similar expressions for V ; too often our theoretical problems included charge distributions that extended to infinity, and the result would be an infinite potential everywhere. Actually, we calculated very few potential fields until the differential form of the potential equation was obtained, $\nabla^2 V = -\rho_v/\epsilon$, or better yet, $\nabla^2 V = 0$. We were then at liberty to select our own zero reference.

The analogous expressions for $\mathbf{A}$ will be derived in the next section, and an example of the calculation of a vector magnetic potential field will be completed.

D7.8. A current sheet, $\mathbf{K} = 2.4\mathbf{a}_z$ A/m, is present at the surface $\rho = 1.2$ in free space. (a) Find $\mathbf{H}$ for $\rho > 1.2$. Find V_m at $P(\rho = 1.5, \phi = 0.6\pi, z = 1)$ if: (b) $V_m = 0$ at $\phi = 0$ and there is a barrier at $\phi = \pi$; (c) $V_m = 0$ at $\phi = 0$ and there is a barrier at $\phi = \pi/2$; (d) $V_m = 0$ at $\phi = \pi$ and there is a barrier at $\phi = 0$; (e) $V_m = 5$ V at $\phi = \pi$ and there is a barrier at $\phi = 0.8\pi$.

Ans. $\frac{2.88}{\rho}\mathbf{a}_\phi$; -5.43 V; 12.7 V; 3.62 V; -9.48 V

D7.9. The value of $\mathbf{A}$ within a solid nonmagnetic conductor of radius a carrying a total current I in the $\mathbf{a}_z$ direction may be found easily. Using the known value of $\mathbf{H}$ or $\mathbf{B}$ for $\rho < a$, then (46) may be solved for $\mathbf{A}$. Select $A = (\mu_0 I \ln 5)/2\pi$ at $\rho = a$ (to correspond with an example in the next section) and find $\mathbf{A}$ at $\rho =$: (a) 0; (b) $0.25a$; (c) $0.75a$; (d) a .

Ans. $0.422I\mathbf{a}_z$ $\mu\text{Wb/m}$; $0.416I\mathbf{a}_z$ $\mu\text{Wb/m}$; $0.366I\mathbf{a}_z$ $\mu\text{Wb/m}$; $0.322I\mathbf{a}_z$ $\mu\text{Wb/m}$

7.7 DERIVATION OF THE STEADY-MAGNETIC-FIELD LAWS

We will now supply the promised proofs of the several relationships between the magnetic field quantities. All these relationships may be obtained from the definitions of $\mathbf{H}$,

$$\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} \quad (3)$$

of $\mathbf{B}$ (in free space),

$$\mathbf{B} = \mu_0 \mathbf{H} \quad (32)$$

and of $\mathbf{A}$,

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (46)$$

Let us first assume that we may express $\mathbf{A}$ by the last equation of Section 7.6,

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu_0 \mathbf{J} dv}{4\pi R} \quad (51)$$

and then demonstrate the correctness of (51) by showing that (3) follows. First, we should add subscripts to indicate the point at which the current element is located (x_1, y_1, z_1) and the point at which $\mathbf{A}$ is given (x_2, y_2, z_2) . The differential volume element dv is then written dv_1 and in rectangular coordinates would be $dx_1 dy_1 dz_1$. The variables of integration are x_1, y_1 , and z_1 . Using these subscripts, then,

$$\mathbf{A}_2 = \int_{\text{vol}} \frac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}} \quad (52)$$

From (32) and (46) we have

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\nabla \times \mathbf{A}}{\mu_0} \quad (53)$$

To show that (3) follows from (52), it is necessary to substitute (52) into (53). This step involves taking the curl of $\mathbf{A}_2$, a quantity expressed in terms of the variables x_2, y_2 , and z_2 , and the curl therefore involves partial derivatives with respect to x_2, y_2 , and z_2 . We do this, placing a subscript on the del operator to remind us of the variables involved in the partial differentiation process,

$$\mathbf{H}_2 = \frac{\nabla_2 \times \mathbf{A}_2}{\mu_0} = \frac{1}{\mu_0} \nabla_2 \times \int_{\text{vol}} \frac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}}$$

The order of partial differentiation and integration is immaterial, and $\mu_0/4\pi$ is constant, allowing us to write

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_{\text{vol}} \nabla_2 \times \frac{\mathbf{J}_1 dv_1}{R_{12}}$$

The curl operation within the integrand represents partial differentiation with respect to x_2, y_2 , and z_2 . The differential volume element dv_1 is a scalar and a function

only of x_1 , y_1 , and z_1 . Consequently, it may be factored out of the curl operation as any other constant, leaving

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_{\text{vol}} \left(\nabla_2 \times \frac{\mathbf{J}_1}{R_{12}} \right) d\nu_1 \quad (54)$$

The curl of the product of a scalar and a vector is given by an identity which may be checked by expansion in rectangular coordinates or obtained from Appendix A.3,

$$\nabla \times (S\mathbf{V}) \equiv (\nabla S) \times \mathbf{V} + S(\nabla \times \mathbf{V}) \quad (55)$$

This identity is used to expand the integrand of (54),

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_{\text{vol}} \left[\left(\nabla_2 \frac{1}{R_{12}} \right) \times \mathbf{J}_1 + \frac{1}{R_{12}} (\nabla_2 \times \mathbf{J}_1) \right] d\nu_1 \quad (56)$$

The second term of this integrand is zero because $\nabla_2 \times \mathbf{J}_1$ indicates partial derivatives of a function of x_1 , y_1 , and z_1 , taken with respect to the variables x_2 , y_2 , and z_2 ; the first set of variables is not a function of the second set, and all partial derivatives are zero.

The first term of the integrand may be determined by expressing R_{12} in terms of the coordinate values,

$$R_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

and taking the gradient of its reciprocal. Problem 7.42 shows that the result is

$$\nabla_2 \frac{1}{R_{12}} = -\frac{\mathbf{R}_{12}}{R_{12}^3} = -\frac{\mathbf{a}_{R12}}{R_{12}^2}$$

Substituting this result into (56), we have

$$\mathbf{H}_2 = -\frac{1}{4\pi} \int_{\text{vol}} \frac{\mathbf{a}_{R12} \times \mathbf{J}_1}{R_{12}^2} d\nu_1$$

or

$$\mathbf{H}_2 = \int_{\text{vol}} \frac{\mathbf{J}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2} d\nu_1$$

which is the equivalent of (3) in terms of current density. Replacing $\mathbf{J}_1 d\nu_1$ by $I_1 d\mathbf{L}_1$, we may rewrite the volume integral as a closed line integral,

$$\mathbf{H}_2 = \oint \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

Equation (51) is therefore correct and agrees with the three definitions (3), (32), and (46).

Next we will prove Ampère's circuital law in point form,

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (28)$$

Combining (28), (32), and (46), we obtain

$$\nabla \times \mathbf{H} = \nabla \times \frac{\mathbf{B}}{\mu_0} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A} \quad (57)$$

We now need the expansion in rectangular coordinates for $\nabla \times \nabla \times \mathbf{A}$. Performing the indicated partial differentiations and collecting the resulting terms, we may write the result as

$$\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (58)$$

where

$$\nabla^2 \mathbf{A} \equiv \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z \quad (59)$$

Equation (59) is the definition (in rectangular coordinates) of the *Laplacian of a vector*.

Substituting (58) into (57), we have

$$\nabla \times \mathbf{H} = \frac{1}{\mu_0} [\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}] \quad (60)$$

and now require expressions for the divergence and the Laplacian of $\mathbf{A}$.

We may find the divergence of $\mathbf{A}$ by applying the divergence operation to (52),

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \nabla_2 \cdot \frac{\mathbf{J}_1}{R_{12}} dv_1 \quad (61)$$

and using the vector identity (44) of Section 4.8,

$$\nabla \cdot (S\mathbf{V}) \equiv \mathbf{V} \cdot (\nabla S) + S(\nabla \cdot \mathbf{V})$$

Thus,

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[\mathbf{J}_1 \cdot \left(\nabla_2 \frac{1}{R_{12}} \right) + \frac{1}{R_{12}} (\nabla_2 \cdot \mathbf{J}_1) \right] dv_1 \quad (62)$$

The second part of the integrand is zero because $\mathbf{J}_1$ is not a function of x_2 , y_2 , and z_2 .

We have already used the result that $\nabla_2(1/R_{12}) = -\mathbf{R}_{12}/R_{12}^3$, and it is just as easily shown that

$$\nabla_1 \frac{1}{R_{12}} = \frac{\mathbf{R}_{12}}{R_{12}^3}$$

or that

$$\nabla_1 \frac{1}{R_{12}} = -\nabla_2 \frac{1}{R_{12}}$$

Equation (62) can therefore be written as

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[-\mathbf{J}_1 \cdot \left(\nabla_1 \frac{1}{R_{12}} \right) \right] dv_1$$

and the vector identity applied again,

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[\frac{1}{R_{12}} (\nabla_1 \cdot \mathbf{J}_1) - \nabla_1 \cdot \left(\frac{\mathbf{J}_1}{R_{12}} \right) \right] dv_1 \quad (63)$$

Because we are concerned only with steady magnetic fields, the continuity equation shows that the first term of (63) is zero. Application of the divergence theorem to the second term gives

$$\nabla_2 \cdot \mathbf{A}_2 = -\frac{\mu_0}{4\pi} \oint_{S_1} \frac{\mathbf{J}_1}{R_{12}} \cdot d\mathbf{S}_1$$

where the surface S_1 encloses the volume throughout which we are integrating. This volume must include all the current, for the original integral expression for $\mathbf{A}$ was an integration such as to include the effect of all the current. Because there is no current outside this volume (otherwise we should have had to increase the volume to include it), we may integrate over a slightly larger volume or a slightly larger enclosing surface without changing $\mathbf{A}$. On this larger surface the current density $\mathbf{J}_1$ must be zero, and therefore the closed surface integral is zero, since the integrand is zero. Hence the divergence of $\mathbf{A}$ is zero.

In order to find the Laplacian of the vector $\mathbf{A}$, let us compare the x component of (51) with the similar expression for electrostatic potential,

$$A_x = \int_{\text{vol}} \frac{\mu_0 J_x dv}{4\pi R} \quad V = \int_{\text{vol}} \frac{\rho_v dv}{4\pi \epsilon_0 R}$$

We note that one expression can be obtained from the other by a straightforward change of variable, J_x for ρ_v , μ_0 for $1/\epsilon_0$, and A_x for V . However, we have derived some additional information about the electrostatic potential which we shall not have to repeat now for the x component of the vector magnetic potential. This takes the form of Poisson's equation,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon_0}$$

which becomes, after the change of variables,

$$\nabla^2 A_x = -\mu_0 J_x$$

Similarly, we have

$$\nabla^2 A_y = -\mu_0 J_y$$

and

$$\nabla^2 A_z = -\mu_0 J_z$$

or

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}} \quad (64)$$

Returning to (60), we can now substitute for the divergence and Laplacian of $\mathbf{A}$ and obtain the desired answer,

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (28)$$

We have already shown the use of Stokes' theorem in obtaining the integral form of Ampère's circuital law from (28) and need not repeat that labor here.

We thus have succeeded in showing that every result we have essentially pulled from thin air¹¹ for magnetic fields follows from the basic definitions of $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{A}$. The derivations are not simple, but they should be understandable on a step-by-step basis.

Finally, let us return to (64) and make use of this formidable second-order vector partial differential equation to find the vector magnetic potential in one simple example. We select the field between conductors of a coaxial cable, with radii of a and b as usual, and current I in the $\mathbf{a}_z$ direction in the inner conductor. Between the conductors, $\mathbf{J} = 0$, and therefore

$$\nabla^2 \mathbf{A} = 0$$

We have already been told (and Problem 7.44 gives us the opportunity to check the results for ourselves) that the vector Laplacian may be expanded as the vector sum of the scalar Laplacians of the three components in rectangular coordinates,

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$

but such a relatively simple result is not possible in other coordinate systems. That is, in cylindrical coordinates, for example,

$$\nabla^2 \mathbf{A} \neq \nabla^2 A_\rho \mathbf{a}_\rho + \nabla^2 A_\phi \mathbf{a}_\phi + \nabla^2 A_z \mathbf{a}_z$$

However, it is not difficult to show for cylindrical coordinates that the z component of the vector Laplacian is the scalar Laplacian of the z component of $\mathbf{A}$, or

$$\nabla^2 \mathbf{A} \Big|_z = \nabla^2 A_z \quad (65)$$

and because the current is entirely in the z direction in this problem, $\mathbf{A}$ has only a z component. Therefore,

$$\nabla^2 A_z = 0$$

or

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial A_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} = 0$$

Thinking symmetrical thoughts about (51) shows us that A_z is a function only of ρ , and thus

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dA_z}{d\rho} \right) = 0$$

We have solved this equation before, and the result is

$$A_z = C_1 \ln \rho + C_2$$

If we choose a zero reference at $\rho = b$, then

$$A_z = C_1 \ln \frac{\rho}{b}$$

¹¹ Free space.

In order to relate C_1 to the sources in our problem, we may take the curl of $\mathbf{A}$,

$$\nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho} \mathbf{a}_\phi = -\frac{C_1}{\rho} \mathbf{a}_\phi = \mathbf{B}$$

obtain $\mathbf{H}$,

$$\mathbf{H} = -\frac{C_1}{\mu_0 \rho} \mathbf{a}_\phi$$

and evaluate the line integral,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_0^{2\pi} -\frac{C_1}{\mu_0 \rho} \mathbf{a}_\phi \cdot \rho d\phi \mathbf{a}_\phi = -\frac{2\pi C_1}{\mu_0}$$

Thus

$$C_1 = -\frac{\mu_0 I}{2\pi}$$

or

$$A_z = \frac{\mu_0 I}{2\pi} \ln \frac{b}{\rho} \quad (66)$$

and

$$H_\phi = \frac{I}{2\pi \rho}$$

as before. A plot of A_z versus ρ for $b = 5a$ is shown in Figure 7.20; the decrease of $|\mathbf{A}|$ with distance from the concentrated current source that the inner conductor represents is evident. The results of Problem D7.9 have also been added to Figure 7.20. The extension of the curve into the outer conductor is left as Problem 7.43.

It is also possible to find A_z between conductors by applying a process some of us informally call “uncurling.” That is, we know $\mathbf{H}$ or $\mathbf{B}$ for the coax, and we may

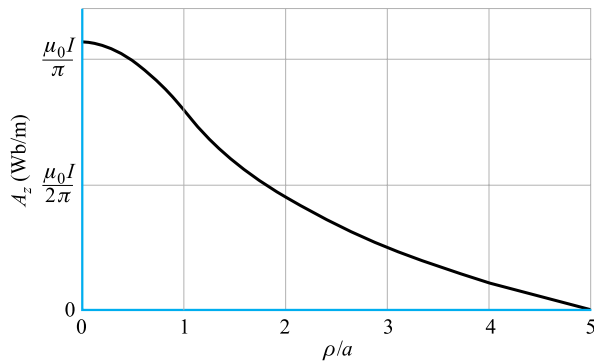


Figure 7.20 The vector magnetic potential is shown within the inner conductor and in the region between conductors for a coaxial cable with $b = 5a$ carrying I in the $\mathbf{a}_z$ direction. $A_z = 0$ is arbitrarily selected at $\rho = b$.

therefore select the ϕ component of $\nabla \times \mathbf{A} = \mathbf{B}$ and integrate to obtain A_z . Try it, you'll like it!

D7.10. Equation (66) is obviously also applicable to the exterior of any conductor of circular cross section carrying a current I in the $\mathbf{a}_z$ direction in free space. The zero reference is arbitrarily set at $\rho = b$. Now consider two conductors, each of 1 cm radius, parallel to the z axis with their axes lying in the $x = 0$ plane. One conductor whose axis is at $(0, 4 \text{ cm}, z)$ carries 12 A in the $\mathbf{a}_z$ direction; the other axis is at $(0, -4 \text{ cm}, z)$ and carries 12 A in the $-\mathbf{a}_z$ direction. Each current has its zero reference for $\mathbf{A}$ located 4 cm from its axis. Find the total $\mathbf{A}$ field at: (a) $(0, 0, z)$; (b) $(0, 8 \text{ cm}, z)$; (c) $(4 \text{ cm}, 4 \text{ cm}, z)$; (d) $(2 \text{ cm}, 4 \text{ cm}, z)$.

Ans. 0; $2.64 \mu\text{Wb/m}$; $1.93 \mu\text{Wb/m}$; $3.40 \mu\text{Wb/m}$

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2. Jordan, E. C., and K. G. Balmain. *Electromagnetic Waves and Radiating Systems*. 2d ed. Englewood Cliffs, N.J.: Prentice-Hall, 1968. Vector magnetic potential is discussed on pp. 90–96.
3. Paul, C. R., K. W. Whites, and S. Y. Nasar. *Introduction to Electromagnetic Fields*. 3d ed. New York: McGraw-Hill, 1998. The vector magnetic potential is presented on pp. 216–20.
4. Skilling, H. H. (See References for Chapter 3.) The “paddle wheel” is introduced on pp. 23–25.

CHAPTER 7 PROBLEMS

- 7.1  (a) Find $\mathbf{H}$ in rectangular components at $P(2, 3, 4)$ if there is a current filament on the z axis carrying 8 mA in the $\mathbf{a}_z$ direction. (b) Repeat if the filament is located at $x = -1, y = 2$. (c) Find $\mathbf{H}$ if both filaments are present.
- 7.2  A filamentary conductor is formed into an equilateral triangle with sides of length ℓ carrying current I . Find the magnetic field intensity at the center of the triangle.
- 7.3  Two semi-infinite filaments on the z axis lie in the regions $-\infty < z < -a$ and $a < z < \infty$. Each carries a current I in the $\mathbf{a}_z$ direction. (a) Calculate $\mathbf{H}$ as a function of ρ and ϕ at $z = 0$. (b) What value of a will cause the magnitude of $\mathbf{H}$ at $\rho = 1, z = 0$, to be one-half the value obtained for an infinite filament?
- 7.4  Two circular current loops are centered on the z axis at $z = \pm h$. Each loop has radius a and carries current I in the $\mathbf{a}_\phi$ direction. (a) Find $\mathbf{H}$ on the z axis over the range $-h < z < h$. Take $I = 1 \text{ A}$ and plot $|\mathbf{H}|$ as a function of z/a if



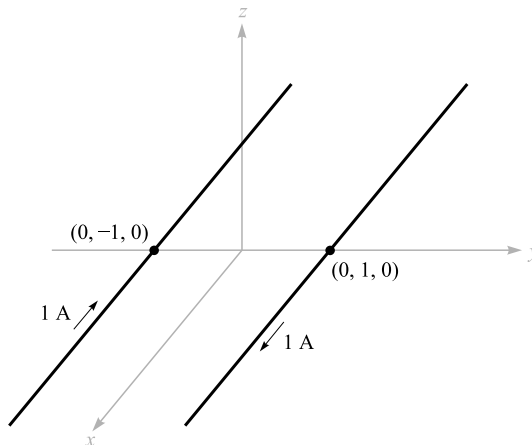


Figure 7.21 See Problem 7.5.

- (b) $h = a/4$; (c) $h = a/2$; (d) $h = a$. Which choice for h gives the most uniform field? These are called Helmholtz coils (of a single turn each in this case), and are used in providing uniform fields.
- 7.5** The parallel filamentary conductors shown in Figure 7.21 lie in free space. Plot $|\mathbf{H}|$ versus y , $-4 < y < 4$, along the line $x = 0$, $z = 2$.
- 7.6** A disk of radius a lies in the xy plane, with the z axis through its center. Surface charge of uniform density ρ_s lies on the disk, which rotates about the z axis at angular velocity Ω rad/s. Find $\mathbf{H}$ at any point on the z axis.
- 7.7** A filamentary conductor carrying current I in the $\mathbf{a}_z$ direction extends along the entire negative z axis. At $z = 0$ it connects to a copper sheet that fills the $x > 0$, $y > 0$ quadrant of the xy plane. (a) Set up the Biot-Savart law and find $\mathbf{H}$ everywhere on the z axis; (b) repeat part (a), but with the copper sheet occupying the *entire* xy plane (Hint: express $\mathbf{a}_\phi$ in terms of $\mathbf{a}_x$ and $\mathbf{a}_y$ and angle ϕ in the integral).
- 7.8** For the finite-length current element on the z axis, as shown in Figure 7.5, use the Biot-Savart law to derive Eq. (9) of Section 7.1.
- 7.9** A current sheet $\mathbf{K} = 8\mathbf{a}_x$ A/m flows in the region $-2 < y < 2$ in the plane $z = 0$. Calculate H at $P(0, 0, 3)$.
- 7.10** A hollow spherical conducting shell of radius a has filamentary connections made at the top ($r = a$, $\theta = 0$) and bottom ($r = a$, $\theta = \pi$). A direct current I flows down the upper filament, down the spherical surface, and out the lower filament. Find $\mathbf{H}$ in spherical coordinates (a) inside and (b) outside the sphere.
- 7.11** An infinite filament on the z axis carries 20π mA in the $\mathbf{a}_z$ direction. Three $\mathbf{a}_z$ -directed uniform cylindrical current sheets are also present: 400 mA/m at