

resistance and capacitance of the structures are related through the simple formula $RC = \epsilon/\sigma$. What basic properties must be true about both the dielectric and the conducting medium for this condition to hold for certain?

- 6.17** Construct a curvilinear-square map for a coaxial capacitor of 3 cm inner radius and 8 cm outer radius. These dimensions are suitable for the drawing. (a) Use your sketch to calculate the capacitance per meter length, assuming $\epsilon_r = 1$. (b) Calculate an exact value for the capacitance per unit length.
- 6.18** Construct a curvilinear-square map of the potential field about two parallel circular cylinders, each of 2.5 cm radius, separated by a center-to-center distance of 13 cm. These dimensions are suitable for the actual sketch if symmetry is considered. As a check, compute the capacitance per meter both from your sketch and from the exact formula. Assume $\epsilon_r = 1$.
- 6.19** Construct a curvilinear-square map of the potential field between two parallel circular cylinders, one of 4 cm radius inside another of 8 cm radius. The two axes are displaced by 2.5 cm. These dimensions are suitable for the drawing. As a check on the accuracy, compute the capacitance per meter from the sketch and from the exact expression:

$$C = \frac{2\pi\epsilon}{\cosh^{-1} [(a^2 + b^2 - D^2)/(2ab)]}$$

where a and b are the conductor radii and D is the axis separation.

- 6.20** A solid conducting cylinder of 4 cm radius is centered within a rectangular conducting cylinder with a 12 cm by 20 cm cross section. (a) Make a full-size sketch of one quadrant of this configuration and construct a curvilinear-square map for its interior. (b) Assume $\epsilon = \epsilon_0$ and estimate C per meter length.
- 6.21** The inner conductor of the transmission line shown in Figure 6.13 has a square cross section $2a \times 2a$, whereas the outer square is $4a \times 5a$. The axes are displaced as shown. (a) Construct a good-sized drawing of this transmission line, say with $a = 2.5$ cm, and then prepare a curvilinear-square plot of the electrostatic field between the conductors. (b) Use the map to calculate the capacitance per meter length if $\epsilon = 1.6\epsilon_0$. (c) How would your result to part (b) change if $a = 0.6$ cm?
- 6.22** Two conducting plates, each 3×6 cm, and three slabs of dielectric, each $1 \times 3 \times 6$ cm, and having dielectric constants of 1, 2, and 3, are assembled into a capacitor with $d = 3$ cm. Determine the two values of capacitance obtained by the two possible methods of assembling the capacitor.
- 6.23** A two-wire transmission line consists of two parallel perfectly conducting cylinders, each having a radius of 0.2 mm, separated by a center-to-center distance of 2 mm. The medium surrounding the wires has $\epsilon_r = 3$ and $\sigma = 1.5$ mS/m. A 100-V battery is connected between the wires. (a) Calculate the magnitude of the charge per meter length on each wire. (b) Using the result of Problem 6.16, find the battery current.

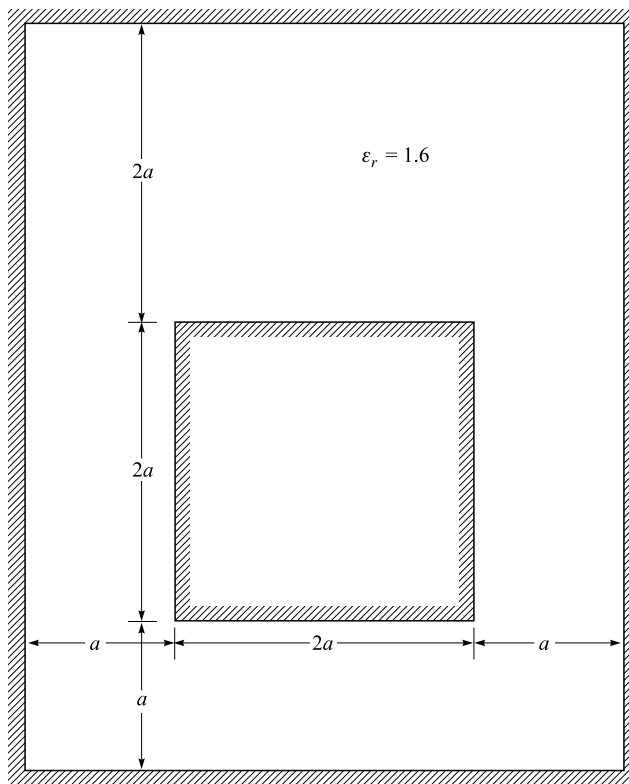


Figure 6.13 See Problem 6.21.

6.24 A potential field in free space is given in spherical coordinates as

$$V(r) = \begin{cases} [\rho_0/(6\epsilon_0)] [3a^2 - r^2] & (r \leq a) \\ (a^3 \rho_0)/(3\epsilon_0 r) & (r \geq a) \end{cases}$$

where ρ_0 and a are constants. (a) Use Poisson's equation to find the volume charge density everywhere. (b) Find the total charge present.

6.25 Let $V = 2xy^2z^3$ and $\epsilon = \epsilon_0$. Given point $P(1, 2, -1)$, find. (a) V at P ; (b) $\mathbf{E}$ at P ; (c) ρ_v at P ; (d) the equation of the equipotential surface passing through P ; (e) the equation of the streamline passing through P . (f) Does V satisfy Laplace's equation?

6.26 Given the spherically symmetric potential field in free space, $V = V_0 e^{-r/a}$, find. (a) ρ_v at $r = a$; (b) the electric field at $r = a$; (c) the total charge.

6.27 Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$.

- 6.28** Show that in a homogeneous medium of conductivity σ , the potential field V satisfies Laplace's equation if any volume charge density present does not vary with time.
- 6.29** Given the potential field $V = (A\rho^4 + B\rho^{-4}) \sin 4\phi$: (a) Show that $\nabla^2 V = 0$. (b) Select A and B so that $V = 100$ V and $|\mathbf{E}| = 500$ V/m at $P(\rho = 1, \phi = 22.5^\circ, z = 2)$.
- 6.30** A parallel-plate capacitor has plates located at $z = 0$ and $z = d$. The region between plates is filled with a material that contains volume charge of uniform density ρ_0 C/m³ and has permittivity ϵ . Both plates are held at ground potential. (a) Determine the potential field between plates. (b) Determine the electric field intensity $\mathbf{E}$ between plates. (c) Repeat parts (a) and (b) for the case of the plate at $z = d$ raised to potential V_0 , with the $z = 0$ plate grounded.
- 6.31** Let $V = (\cos 2\phi)/\rho$ in free space. (a) Find the volume charge density at point $A(0.5, 60^\circ, 1)$. (b) Find the surface charge density on a conductor surface passing through the point $B(2, 30^\circ, 1)$.
- 6.32** A uniform volume charge has constant density $\rho_v = \rho_0$ C/m³ and fills the region $r < a$, in which permittivity ϵ is assumed. A conducting spherical shell is located at $r = a$ and is held at ground potential. Find (a) the potential everywhere; (b) the electric field intensity, $\mathbf{E}$, everywhere.
- 6.33** The functions $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$ both satisfy Laplace's equation in the region $a < \rho < b$, $0 \leq \phi < 2\pi$, $-L < z < L$; each is zero on the surfaces $\rho = b$ for $-L < z < L$; $z = -L$ for $a < \rho < b$; and $z = L$ for $a < \rho < b$; and each is 100 V on the surface $\rho = a$ for $-L < z < L$. (a) In the region specified, is Laplace's equation satisfied by the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? (b) On the boundary surfaces specified, are the potential values given in this problem obtained from the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? (c) Are the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$ identical with V_1 ?
- 6.34** Consider the parallel-plate capacitor of Problem 6.30, but this time the charged dielectric exists only between $z = 0$ and $z = b$, where $b < d$. Free space fills the region $b < z < d$. Both plates are at ground potential. By solving Laplace's and Poisson's equations, find (a) $V(z)$ for $0 < z < d$; (b) the electric field intensity for $0 < z < d$. No surface charge exists at $z = b$, so both V and $\mathbf{D}$ are continuous there.
- 6.35** The conducting planes $2x + 3y = 12$ and $2x + 3y = 18$ are at potentials of 100 V and 0, respectively. Let $\epsilon = \epsilon_0$ and find (a) V at $P(5, 2, 6)$; (b) $\mathbf{E}$ at P .
- 6.36** The derivation of Laplace's and Poisson's equations assumed constant permittivity, but there are cases of spatially varying permittivity in which the equations will still apply. Consider the vector identity, $\nabla \cdot (\psi \mathbf{G}) = \mathbf{G} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{G}$, where ψ and $\mathbf{G}$ are scalar and vector functions, respectively.

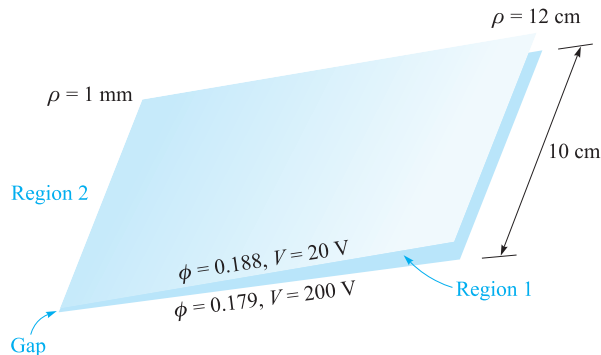


Figure 6.14 See Problem 6.39.

Determine a general rule on the allowed *directions* in which ϵ may vary with respect to the local electric field.

- 6.37** Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100 V and the outer at 0 V, find (a) the location of the 20 V equipotential surface; (b) $E_{\rho \max}$; (c) ϵ_r if the charge per meter length on the inner cylinder is 20 nC/m.
- 6.38** Repeat Problem 6.37, but with the dielectric only partially filling the volume, within $0 < \phi < \pi$, and with free space in the remaining volume.
- 6.39** The two conducting planes illustrated in Figure 6.14 are defined by $0.001 < \rho < 0.120$ m, $0 < z < 0.1$ m, $\phi = 0.179$ and 0.188 rad. The medium surrounding the planes is air. For Region 1, $0.179 < \phi < 0.188$; neglect fringing and find (a) $V(\phi)$; (b) $\mathbf{E}(\rho)$; (c) $\mathbf{D}(\rho)$; (d) ρ_s on the upper surface of the lower plane; (e) Q on the upper surface of the lower plane. (f) Repeat parts (a) through (c) for Region 2 by letting the location of the upper plane be $\phi = 0.188 - 2\pi$, and then find ρ_s and Q on the lower surface of the lower plane. (g) Find the total charge on the lower plane and the capacitance between the planes.
- 6.40** A parallel-plate capacitor is made using two circular plates of radius a , with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho^2/a^2)$. Find (a) $V(z)$; (b) $\mathbf{E}$; (c) Q ; (d) C . This is a reprise of Problem 6.8, but it starts with Laplace's equation.
- 6.41** Concentric conducting spheres are located at $r = 5$ mm and $r = 20$ mm. The region between the spheres is filled with a perfect dielectric. If the inner sphere is at 100 V and the outer sphere is at 0 V (a) Find the

location of the 20 V equipotential surface. (b) Find $E_{r,\max}$. (c) Find ϵ_r if the surface charge density on the inner sphere is $1.0 \mu\text{C}/\text{m}^2$.

- 6.42** The hemisphere $0 < r < a$, $0 < \theta < \pi/2$, is composed of homogeneous conducting material of conductivity σ . The flat side of the hemisphere rests on a perfectly conducting plane. Now, the material within the conical region $0 < \theta < \alpha$, $0 < r < a$ is drilled out and replaced with material that is perfectly conducting. An air gap is maintained between the $r = 0$ tip of this new material and the plane. What resistance is measured between the two perfect conductors? Neglect fringing fields.
- 6.43** Two coaxial conducting cones have their vertices at the origin and the z axis as their axis. Cone A has the point $A(1, 0, 2)$ on its surface, while cone B has the point $B(0, 3, 2)$ on its surface. Let $V_A = 100$ V and $V_B = 20$ V. Find (a) α for each cone; (b) V at $P(1, 1, 1)$.
- 6.44** A potential field in free space is given as $V = 100 \ln \tan(\theta/2) + 50$ V. (a) Find the maximum value of $|\mathbf{E}_\theta|$ on the surface $\theta = 40^\circ$ for $0.1 < r < 0.8$ m, $60^\circ < \phi < 90^\circ$. (b) Describe the surface $V = 80$ V.
- 6.45** In free space, let $\rho_v = 200\epsilon_0/r^{2.4}$. (a) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2 E_r \rightarrow 0$ when $r \rightarrow 0$, and also that $V \rightarrow 0$ as $r \rightarrow \infty$. (b) Now find $V(r)$ by using Gauss's law and a line integral.
- 6.46** By appropriate solution of Laplace's and Poisson's equations, determine the absolute potential at the center of a sphere of radius a , containing uniform volume charge of density ρ_0 . Assume permittivity ϵ_0 everywhere. *Hint:* What must be true about the potential and the electric field at $r = 0$ and at $r = a$?

The Steady Magnetic Field

At this point, the concept of a field should be a familiar one. Since we first accepted the experimental law of forces existing between two point charges and defined electric field intensity as the force per unit charge on a test charge in the presence of a second charge, we have discussed numerous fields. These fields possess no real physical basis, for physical measurements must always be in terms of the forces on the charges in the detection equipment. Those charges that are the source cause measurable forces to be exerted on other charges, which we may think of as detector charges. The fact that we attribute a field to the source charges and then determine the effect of this field on the detector charges amounts merely to a division of the basic problem into two parts for convenience.

We will begin our study of the magnetic field with a definition of the magnetic field itself and show how it arises from a current distribution. The effect of this field on other currents, or the second half of the physical problem, will be discussed in Chapter 8. As we did with the electric field, we confine our initial discussion to free-space conditions, and the effect of material media will also be saved for discussion in Chapter 8.

The relation of the steady magnetic field to its source is more complicated than is the relation of the electrostatic field to its source. We will find it necessary to accept several laws temporarily on faith alone. The proof of the laws does exist and is available on the Web site for the disbelievers or the more advanced student. ■

7.1 BIOT-SAVART LAW

The source of the steady magnetic field may be a permanent magnet, an electric field changing linearly with time, or a direct current. We will largely ignore the permanent magnet and save the time-varying electric field for a later discussion. Our present study will concern the magnetic field produced by a differential dc element in free space.

We may think of this differential current element as a vanishingly small section of a current-carrying filamentary conductor, where a filamentary conductor is the limiting

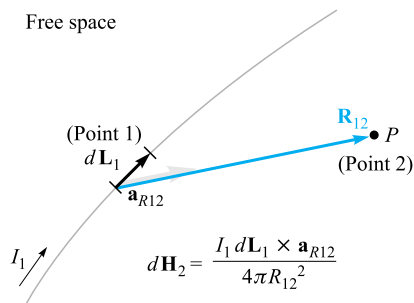


Figure 7.1 The law of Biot-Savart expresses the magnetic field intensity $d\mathbf{H}_2$ produced by a differential current element $I_1 d\mathbf{L}_1$. The direction of $d\mathbf{H}_2$ is into the page.

case of a cylindrical conductor of circular cross section as the radius approaches zero. We assume a current I flowing in a differential vector length of the filament $d\mathbf{L}$. The law of Biot-Savart¹ then states that at any point P the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired; also, the magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point P . The direction of the magnetic field intensity is normal to the plane containing the differential filament and the line drawn from the filament to the point P . Of the two possible normals, that one to be chosen is the one which is in the direction of progress of a right-handed screw turned from $d\mathbf{L}$ through the smaller angle to the line from the filament to P . Using rationalized mks units, the constant of proportionality is $1/4\pi$.

The *Biot-Savart law*, just described in some 150 words, may be written concisely using vector notation as

$$d\mathbf{H} = \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I d\mathbf{L} \times \mathbf{R}}{4\pi R^3} \quad (1)$$

The units of the *magnetic field intensity* $\mathbf{H}$ are evidently amperes per meter (A/m). The geometry is illustrated in Figure 7.1. Subscripts may be used to indicate the point to which each of the quantities in (1) refers. If we locate the current element at point 1 and describe the point P at which the field is to be determined as point 2, then

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2} \quad (2)$$

¹ Biot and Savart were colleagues of Ampère, and all three were professors of physics at the Collège de France at one time or another. The Biot-Savart law was proposed in 1820.

The law of Biot-Savart is sometimes called *Ampère's law for the current element*, but we will retain the former name because of possible confusion with Ampère's circuital law, to be discussed later.

In some aspects, the Biot-Savart law is reminiscent of Coulomb's law when that law is written for a differential element of charge,

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi\epsilon_0 R_{12}^2}$$

Both show an inverse-square-law dependence on distance, and both show a linear relationship between source and field. The chief difference appears in the direction of the field.

It is impossible to check experimentally the law of Biot-Savart as expressed by (1) or (2) because the differential current element cannot be isolated. We have restricted our attention to direct currents only, so the charge density is not a function of time. The continuity equation in Section 5.2, Eq. (5),

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

therefore shows that

$$\nabla \cdot \mathbf{J} = 0$$

or upon applying the divergence theorem,

$$\oint_s \mathbf{J} \cdot d\mathbf{S} = 0$$

The total current crossing any closed surface is zero, and this condition may be satisfied only by assuming a current flow around a closed path. It is this current flowing in a closed circuit that must be our experimental source, not the differential element.

It follows that only the integral form of the Biot-Savart law can be verified experimentally,

$$\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} \quad (3)$$

Equation (1) or (2), of course, leads directly to the integral form (3), but other differential expressions also yield the same integral formulation. Any term may be added to (1) whose integral around a closed path is zero. That is, any conservative field could be added to (1). The gradient of any scalar field always yields a conservative field, and we could therefore add a term ∇G to (1), where G is a general scalar field, without changing (3) in the slightest. This qualification on (1) or (2) is mentioned to show that if we later ask some foolish questions, not subject to any experimental check, concerning the force exerted by one *differential* current element on another, we should expect foolish answers.

The Biot-Savart law may also be expressed in terms of distributed sources, such as current density $\mathbf{J}$ and *surface current density* $\mathbf{K}$. Surface current flows in a sheet of vanishingly small thickness, and the current density $\mathbf{J}$, measured in amperes per square

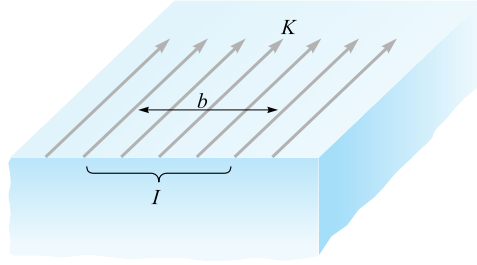


Figure 7.2 The total current I within a transverse width b , in which there is a *uniform* surface current density K , is Kb .

meter, is therefore infinite. Surface current density, however, is measured in amperes per meter width and designated by $\mathbf{K}$. If the surface current density is uniform, the total current I in any width b is

$$I = Kb$$

where we assume that the width b is measured perpendicularly to the direction in which the current is flowing. The geometry is illustrated by Figure 7.2. For a nonuniform surface current density, integration is necessary:

$$I = \int K dN \quad (4)$$

where dN is a differential element of the path *across* which the current is flowing. Thus the differential current element $I d\mathbf{L}$, where $d\mathbf{L}$ is in the direction of the current, may be expressed in terms of surface current density $\mathbf{K}$ or current density $\mathbf{J}$,

$$I d\mathbf{L} = \mathbf{K} dS = \mathbf{J} dv \quad (5)$$

and alternate forms of the Biot-Savart law obtained,

$$\mathbf{H} = \int_s \frac{\mathbf{K} \times \mathbf{a}_R dS}{4\pi R^2} \quad (6)$$

and

$$\mathbf{H} = \int_{\text{vol}} \frac{\mathbf{J} \times \mathbf{a}_R dv}{4\pi R^2} \quad (7)$$

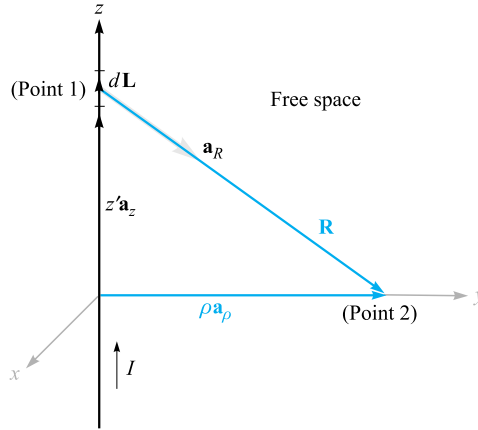


Figure 7.3 An infinitely long straight filament carrying a direct current I . The field at point 2 is $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$.

We illustrate the application of the Biot-Savart law by considering an infinitely long straight filament. We apply (2) first and then integrate. This, of course, is the same as using the integral form (3) in the first place.²

Referring to Figure 7.3, we should recognize the symmetry of this field. No variation with z or with ϕ can exist. Point 2, at which we will determine the field, is therefore chosen in the $z = 0$ plane. The field point $\mathbf{r}$ is therefore $r = \rho\mathbf{a}_\rho$. The source point $\mathbf{r}'$ is given by $\mathbf{r}' = z'\mathbf{a}_z$, and therefore

$$\mathbf{R}_{12} = \mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

so that

$$\mathbf{a}_{R12} = \frac{\rho\mathbf{a}_\rho - z'\mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

We take $d\mathbf{L} = dz'\mathbf{a}_z$ and (2) becomes

$$d\mathbf{H}_2 = \frac{I dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Because the current is directed toward increasing values of z' , the limits are $-\infty$ and ∞ on the integral, and we have

$$\begin{aligned} \mathbf{H}_2 &= \int_{-\infty}^{\infty} \frac{I dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}} \\ &= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}} \end{aligned}$$

² The closed path for the current may be considered to include a return filament parallel to the first filament and infinitely far removed. An outer coaxial conductor of infinite radius is another theoretical possibility. Practically, the problem is an impossible one, but we should realize that our answer will be quite accurate near a very long, straight wire having a distant return path for the current.

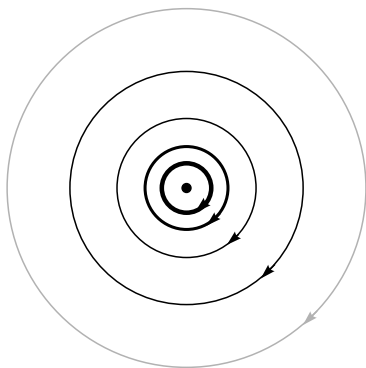


Figure 7.4 The streamlines of the magnetic field intensity about an infinitely long straight filament carrying a direct current I . The direction of I is into the page.

At this point the unit vector $\mathbf{a}_\phi$ under the integral sign should be investigated, for it is not always a constant, as are the unit vectors of the rectangular coordinate system. A vector is constant when its magnitude and direction are both constant. The unit vector certainly has constant magnitude, but its direction may change. Here $\mathbf{a}_\phi$ changes with the coordinate ϕ but not with ρ or z . Fortunately, the integration here is with respect to z' , and $\mathbf{a}_\phi$ is a constant and may be removed from under the integral sign,



$$\begin{aligned}\mathbf{H}_2 &= \frac{I\rho\mathbf{a}_\phi}{4\pi} \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}} \\ &= \frac{I\rho\mathbf{a}_\phi}{4\pi} \left. \frac{z'}{\rho^2\sqrt{\rho^2 + z'^2}} \right|_{-\infty}^{\infty}\end{aligned}$$

and

$$\mathbf{H}_2 = \frac{I}{2\pi\rho} \mathbf{a}_\phi \quad (8)$$

The magnitude of the field is not a function of ϕ or z , and it varies inversely with the distance from the filament. The direction of the magnetic-field-intensity vector is circumferential. The streamlines are therefore circles about the filament, and the field may be mapped in cross section as in Figure 7.4.

The separation of the streamlines is proportional to the radius, or inversely proportional to the magnitude of $\mathbf{H}$. To be specific, the streamlines have been drawn with curvilinear squares in mind. As yet, we have no name for the family of lines³ that are perpendicular to these circular streamlines, but the spacing of the streamlines has

³ If you can't wait, see Section 7.6.

been adjusted so that the addition of this second set of lines will produce an array of curvilinear squares.

A comparison of Figure 7.4 with the map of the *electric* field about an infinite line *charge* shows that the streamlines of the magnetic field correspond exactly to the equipotentials of the electric field, and the unnamed (and undrawn) perpendicular family of lines in the magnetic field corresponds to the streamlines of the electric field. This correspondence is not an accident, but there are several other concepts which must be mastered before the analogy between electric and magnetic fields can be explored more thoroughly.

Using the Biot-Savart law to find $\mathbf{H}$ is in many respects similar to the use of Coulomb's law to find $\mathbf{E}$. Each requires the determination of a moderately complicated integrand containing vector quantities, followed by an integration. When we were concerned with Coulomb's law we solved a number of examples, including the fields of the point charge, line charge, and sheet of charge. The law of Biot-Savart can be used to solve analogous problems in magnetic fields, and some of these problems appear as exercises at the end of the chapter rather than as examples here.

One useful result is the field of the finite-length current element, shown in Figure 7.5. It turns out (see Problem 7.8 at the end of the chapter) that $\mathbf{H}$ is most easily expressed in terms of the angles α_1 and α_2 , as identified in the figure. The result is

$$\mathbf{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi \quad (9)$$

If one or both ends are below point 2, then α_1 is or both α_1 and α_2 are negative.

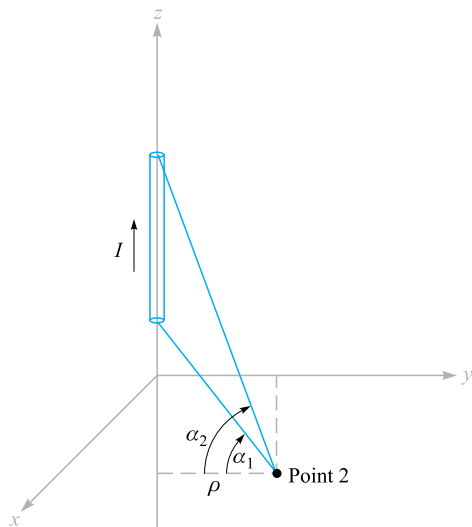


Figure 7.5 The magnetic field intensity caused by a finite-length current filament on the z axis is $(I/4\pi\rho)(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$.

Equation (9) may be used to find the magnetic field intensity caused by current filaments arranged as a sequence of straight-line segments.

EXAMPLE 7.1

As a numerical example illustrating the use of (9), we determine $\mathbf{H}$ at $P_2(0.4, 0.3, 0)$ in the field of an 8-A filamentary current is directed inward from infinity to the origin on the positive x axis, and then outward to infinity along the y axis. This arrangement is shown in Figure 7.6.

Solution. We first consider the semi-infinite current on the x axis, identifying the two angles, $\alpha_{1x} = -90^\circ$ and $\alpha_{2x} = \tan^{-1}(0.4/0.3) = 53.1^\circ$. The radial distance ρ is measured from the x axis, and we have $\rho_x = 0.3$. Thus, this contribution to $\mathbf{H}_2$ is

$$\mathbf{H}_{2(x)} = \frac{8}{4\pi(0.3)}(\sin 53.1^\circ + 1)\mathbf{a}_\phi = \frac{2}{0.3\pi}(1.8)\mathbf{a}_\phi = \frac{12}{\pi}\mathbf{a}_\phi$$

The unit vector $\mathbf{a}_\phi$ must also be referred to the x axis. We see that it becomes $-\mathbf{a}_z$. Therefore,

$$\mathbf{H}_{2(x)} = -\frac{12}{\pi}\mathbf{a}_z \text{ A/m}$$

For the current on the y axis, we have $\alpha_{1y} = -\tan^{-1}(0.3/0.4) = -36.9^\circ$, $\alpha_{2y} = 90^\circ$, and $\rho_y = 0.4$. It follows that

$$\mathbf{H}_{2(y)} = \frac{8}{4\pi(0.4)}(1 + \sin 36.9^\circ)(-\mathbf{a}_z) = -\frac{8}{\pi}\mathbf{a}_z \text{ A/m}$$

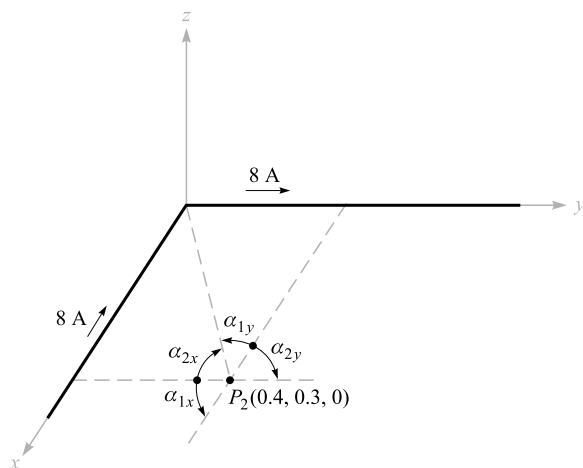


Figure 7.6 The individual fields of two semi-infinite current segments are found by (9) and added to obtain $\mathbf{H}_2$ at P_2 .

Adding these results, we have

$$\mathbf{H}_2 = \mathbf{H}_{2(x)} + \mathbf{H}_{2(y)} = -\frac{20}{\pi} \mathbf{a}_z = -6.37 \mathbf{a}_z \text{ A/m}$$

D7.1. Given the following values for P_1 , P_2 , and $I_1 \Delta L_1$, calculate $\Delta \mathbf{H}_2$:
 (a) $P_1(0, 0, 2)$, $P_2(4, 2, 0)$, $2\pi \mathbf{a}_z \mu\text{A}\cdot\text{m}$; (b) $P_1(0, 2, 0)$, $P_2(4, 2, 3)$, $2\pi \mathbf{a}_z \mu\text{A}\cdot\text{m}$;
 (c) $P_1(1, 2, 3)$, $P_2(-3, -1, 2)$, $2\pi(-\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z) \mu\text{A}\cdot\text{m}$.

Ans. $-8.51 \mathbf{a}_x + 17.01 \mathbf{a}_y \text{ nA/m}$; $16 \mathbf{a}_y \text{ nA/m}$; $18.9 \mathbf{a}_x - 33.9 \mathbf{a}_y + 26.4 \mathbf{a}_z \text{ nA/m}$

D7.2. A current filament carrying 15 A in the $\mathbf{a}_z$ direction lies along the entire z axis. Find $\mathbf{H}$ in rectangular coordinates at: (a) $P_A(\sqrt{20}, 0, 4)$; (b) $P_B(2, -4, 4)$.

Ans. $0.534 \mathbf{a}_y \text{ A/m}$; $0.477 \mathbf{a}_x + 0.239 \mathbf{a}_y \text{ A/m}$

7.2 AMPÈRE'S CIRCUITAL LAW

After solving a number of simple electrostatic problems with Coulomb's law, we found that the same problems could be solved much more easily by using Gauss's law whenever a high degree of symmetry was present. Again, an analogous procedure exists in magnetic fields. Here, the law that helps us solve problems more easily is known as *Ampère's circuital⁴ law*, sometimes called Ampère's work law. This law may be derived from the Biot-Savart law (see Section 7.7).

Ampère's circuital law states that the line integral of $\mathbf{H}$ about any *closed* path is exactly equal to the direct current enclosed by that path,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I \quad (10)$$

We define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed.

Referring to Figure 7.7, which shows a circular wire carrying a direct current I , the line integral of $\mathbf{H}$ about the closed paths lettered a and b results in an answer of I ; the integral about the closed path c which passes through the conductor gives an answer less than I and is exactly that portion of the total current that is enclosed by the path c . Although paths a and b give the same answer, the integrands are, of course, different. The line integral directs us to multiply the component of $\mathbf{H}$ in the direction of the path by a small increment of path length at one point of the path, move along the path to the next incremental length, and repeat the process, continuing until the path is completely traversed. Because $\mathbf{H}$ will generally vary from point to point, and because paths a and b are not alike, the contributions to the integral made by, say,

⁴ The preferred pronunciation puts the accent on "circ-."

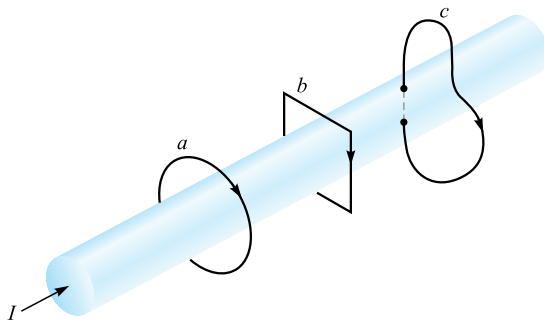


Figure 7.7 A conductor has a total current I . The line integral of $\mathbf{H}$ about the closed paths a and b is equal to I , and the integral around path c is less than I , since the entire current is not enclosed by the path.

each micrometer of path length are quite different. Only the final answers are the same.

We should also consider exactly what is meant by the expression “current enclosed by the path.” Suppose we solder a circuit together after passing the conductor once through a rubber band, which we use to represent the closed path. Some strange and formidable paths can be constructed by twisting and knotting the rubber band, but if neither the rubber band nor the conducting circuit is broken, the current enclosed by the path is that carried by the conductor. Now replace the rubber band by a circular ring of spring steel across which is stretched a rubber sheet. The steel loop forms the closed path, and the current-carrying conductor must pierce the rubber sheet if the current is to be enclosed by the path. Again, we may twist the steel loop, and we may also deform the rubber sheet by pushing our fist into it or folding it in any way we wish. A single current-carrying conductor still pierces the sheet once, and this is the true measure of the current enclosed by the path. If we should thread the conductor once through the sheet from front to back and once from back to front, the total current enclosed by the path is the algebraic sum, which is zero.

In more general language, given a closed path, we recognize this path as the perimeter of an infinite number of surfaces (not closed surfaces). Any current-carrying conductor enclosed by the path must pass through every one of these surfaces once. Certainly some of the surfaces may be chosen in such a way that the conductor pierces them twice in one direction and once in the other direction, but the algebraic total current is still the same.

We will find that the nature of the closed path is usually extremely simple and can be drawn on a plane. The simplest surface is, then, that portion of the plane enclosed by the path. We need merely find the total current passing through this region of the plane.

The application of Gauss’s law involves finding the total charge enclosed by a closed surface; the application of Ampère’s circuital law involves finding the total current enclosed by a closed path.

Let us again find the magnetic field intensity produced by an infinitely long filament carrying a current I . The filament lies on the z axis in free space (as in Figure 7.3), and the current flows in the direction given by $\mathbf{a}_z$. Symmetry inspection comes first, showing that there is no variation with z or ϕ . Next we determine which components of $\mathbf{H}$ are present by using the Biot-Savart law. Without specifically using the cross product, we may say that the direction of $d\mathbf{H}$ is perpendicular to the plane containing $d\mathbf{L}$ and $\mathbf{R}$ and therefore is in the direction of $\mathbf{a}_\phi$. Hence the only component of $\mathbf{H}$ is H_ϕ , and it is a function only of ρ .

We therefore choose a path, to any section of which $\mathbf{H}$ is either perpendicular or tangential, and along which H is constant. The first requirement (perpendicularity or tangency) allows us to replace the dot product of Ampère's circuital law with the product of the scalar magnitudes, except along that portion of the path where $\mathbf{H}$ is normal to the path and the dot product is zero; the second requirement (constancy) then permits us to remove the magnetic field intensity from the integral sign. The integration required is usually trivial and consists of finding the length of that portion of the path to which $\mathbf{H}$ is parallel.

In our example, the path must be a circle of radius ρ , and Ampère's circuital law becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi \rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho}$$

as before.

As a second example of the application of Ampère's circuital law, consider an infinitely long coaxial transmission line carrying a uniformly distributed total current I in the center conductor and $-I$ in the outer conductor. The line is shown in Figure 7.8a. Symmetry shows that H is not a function of ϕ or z . In order to determine the components present, we may use the results of the previous example by considering the solid conductors as being composed of a large number of filaments. No filament has a z component of $\mathbf{H}$. Furthermore, the H_ρ component at $\phi = 0^\circ$, produced by one filament located at $\rho = \rho_1$, $\phi = \phi_1$, is canceled by the H_ρ component produced by a symmetrically located filament at $\rho = \rho_1$, $\phi = -\phi_1$. This symmetry is illustrated by Figure 7.8b. Again we find only an H_ϕ component which varies with ρ .

A circular path of radius ρ , where ρ is larger than the radius of the inner conductor but less than the inner radius of the outer conductor, then leads immediately to

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

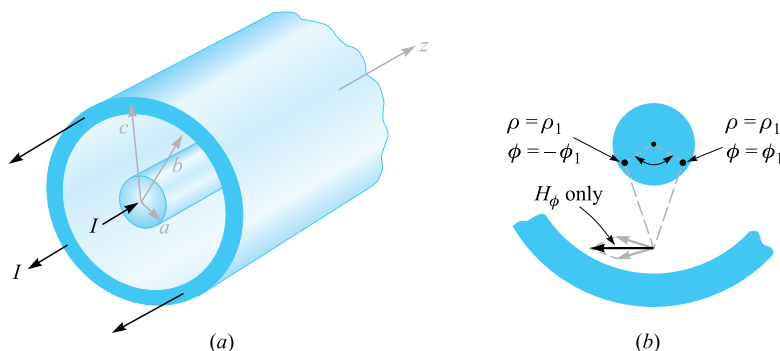


Figure 7.8 (a) Cross section of a coaxial cable carrying a uniformly distributed current I in the inner conductor and $-I$ in the outer conductor. The magnetic field at any point is most easily determined by applying Ampère's circuital law about a circular path. (b) Current filaments at $\rho = \rho_1$, $\phi = \pm\phi_1$, produces H_ρ components which cancel. For the total field, $\mathbf{H} = H_\phi \mathbf{a}_\phi$.

If we choose ρ smaller than the radius of the inner conductor, the current enclosed is

$$I_{\text{encl}} = I \frac{\rho^2}{a^2}$$

and

$$2\pi\rho H_\phi = I \frac{\rho^2}{a^2}$$

or

$$H_\phi = \frac{I\rho}{2\pi a^2} \quad (\rho < a)$$

If the radius ρ is larger than the outer radius of the outer conductor, no current is enclosed and

$$H_\phi = 0 \quad (\rho > c)$$

Finally, if the path lies within the outer conductor, we have

$$2\pi\rho H_\phi = I - I \left(\frac{\rho^2 - b^2}{c^2 - b^2} \right)$$

$$H_\phi = \frac{I}{2\pi\rho} \frac{c^2 - \rho^2}{c^2 - b^2} \quad (b < \rho < c)$$

The magnetic-field-strength variation with radius is shown in Figure 7.9 for a coaxial cable in which $b = 3a$, $c = 4a$. It should be noted that the magnetic field intensity $\mathbf{H}$ is continuous at all the conductor boundaries. In other words, a slight increase in the radius of the closed path does not result in the enclosure of a tremendously different current. The value of H_ϕ shows no sudden jumps.



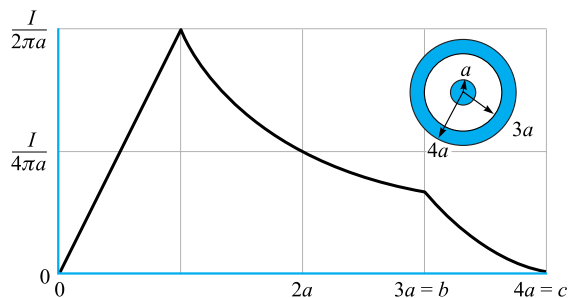


Figure 7.9 The magnetic field intensity as a function of radius in an infinitely long coaxial transmission line with the dimensions shown.

The external field is zero. This, we see, results from equal positive and negative currents enclosed by the path. Each produces an external field of magnitude $I/2\pi\rho$, but complete cancellation occurs. This is another example of “shielding”; such a coaxial cable carrying large currents would, in principle, not produce any noticeable effect in an adjacent circuit.

As a final example, let us consider a sheet of current flowing in the positive y direction and located in the $z = 0$ plane. We may think of the return current as equally divided between two distant sheets on either side of the sheet we are considering. A sheet of uniform surface current density $\mathbf{K} = K_y \mathbf{a}_y$ is shown in Figure 7.10. $\mathbf{H}$ cannot vary with x or y . If the sheet is subdivided into a number of filaments, it is evident that no filament can produce an H_y component. Moreover, the Biot-Savart law shows that the contributions to H_z produced by a symmetrically located pair of filaments cancel. Thus, H_z is zero also; only an H_x component is present. We therefore choose the path 1-1'-2'-2-1 composed of straight-line segments that are either parallel or

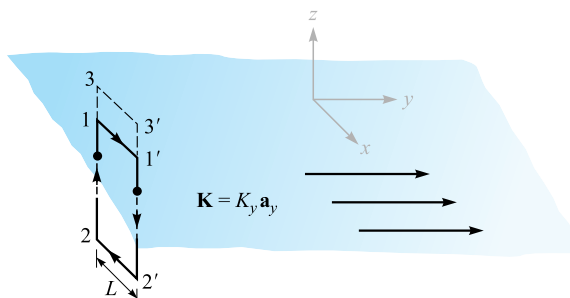


Figure 7.10 A uniform sheet of surface current $\mathbf{K} = K_y \mathbf{a}_y$ in the $z = 0$ plane. $\mathbf{H}$ may be found by applying Ampère's circuital law about the paths 1-1'-2'-2-1 and 3-3'-2'-2-3.

perpendicular to H_x . Ampère's circuital law gives

$$H_{x1}L + H_{x2}(-L) = K_y L$$

or

$$H_{x1} - H_{x2} = K_y$$

If the path 3-3'-2'-2-3 is now chosen, the same current is enclosed, and

$$H_{x3} - H_{x2} = K_y$$

and therefore

$$H_{x3} = H_{x1}$$

It follows that H_x is the same for all positive z . Similarly, H_x is the same for all negative z . Because of the symmetry, then, the magnetic field intensity on one side of the current sheet is the negative of that on the other. Above the sheet,

$$H_x = \frac{1}{2}K_y \quad (z > 0)$$

while below it

$$H_x = -\frac{1}{2}K_y \quad (z < 0)$$

Letting $\mathbf{a}_N$ be a unit vector normal (outward) to the current sheet, the result may be written in a form correct for all z as

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N \quad (11)$$

If a second sheet of current flowing in the opposite direction, $\mathbf{K} = -K_y\mathbf{a}_y$, is placed at $z = h$, (11) shows that the field in the region between the current sheets is

$$\mathbf{H} = \mathbf{K} \times \mathbf{a}_N \quad (0 < z < h) \quad (12)$$

and is zero elsewhere,

$$\mathbf{H} = 0 \quad (z < 0, z > h) \quad (13)$$

The most difficult part of the application of Ampère's circuital law is the determination of the components of the field that are present. The surest method is the logical application of the Biot-Savart law and a knowledge of the magnetic fields of simple form.

Problem 7.13 at the end of this chapter outlines the steps involved in applying Ampère's circuital law to an infinitely long solenoid of radius a and uniform current density $K_a\mathbf{a}_\phi$, as shown in Figure 7.11a. For reference, the result is

$$\mathbf{H} = K_a\mathbf{a}_z \quad (\rho < a) \quad (14a)$$

$$\mathbf{H} = 0 \quad (\rho > a) \quad (14b)$$

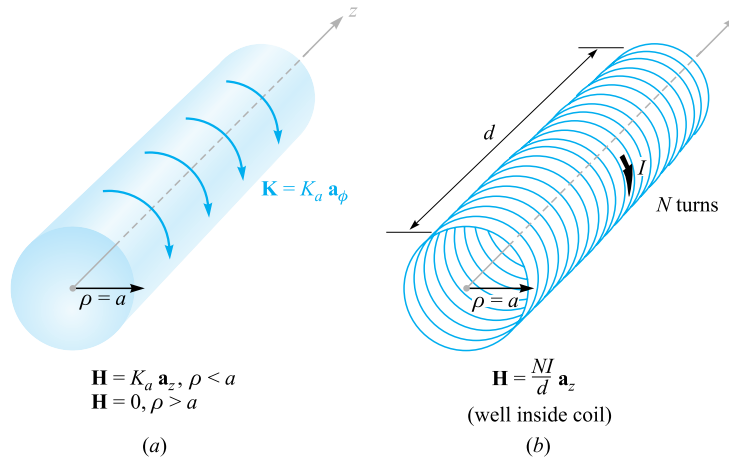


Figure 7.11 (a) An ideal solenoid of infinite length with a circular current sheet $\mathbf{K} = K_a \mathbf{a}_\phi$. (b) An N -turn solenoid of finite length d .

If the solenoid has a finite length d and consists of N closely wound turns of a filament that carries a current I (Figure 7.11b), then the field at points well within the solenoid is given closely by

$$\mathbf{H} = \frac{NI}{d} \mathbf{a}_z \quad (\text{well within the solenoid}) \quad (15)$$

The approximation is useful if it is not applied closer than two radii to the open ends, nor closer to the solenoid surface than twice the separation between turns.

For the toroids shown in Figure 7.12, it can be shown that the magnetic field intensity for the ideal case, Figure 7.12a, is

$$\mathbf{H} = K_a \frac{\rho_0 - a}{\rho} \mathbf{a}_\phi \quad (\text{inside toroid}) \quad (16a)$$

$$\mathbf{H} = 0 \quad (\text{outside}) \quad (16b)$$

For the N -turn toroid of Figure 7.12b, we have the good approximations,

$$\mathbf{H} = \frac{NI}{2\pi\rho} \mathbf{a}_\phi \quad (\text{inside toroid}) \quad (17a)$$

$$\mathbf{H} = 0 \quad (\text{outside}) \quad (17b)$$

as long as we consider points removed from the toroidal surface by several times the separation between turns.

Toroids having rectangular cross sections are also treated quite readily, as you can see for yourself by trying Problem 7.14.

Accurate formulas for solenoids, toroids, and coils of other shapes are available in Section 2 of the *Standard Handbook for Electrical Engineers* (see References for Chapter 5).

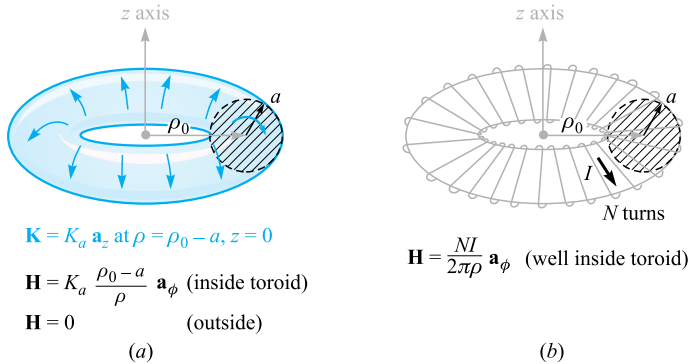


Figure 7.12 (a) An ideal toroid carrying a surface current $\mathbf{K}$ in the direction shown. (b) An N -turn toroid carrying a filamentary current I .

D7.3. Express the value of $\mathbf{H}$ in rectangular components at $P(0, 0.2, 0)$ in the field of: (a) a current filament, 2.5 A in the $\mathbf{a}_z$ direction at $x = 0.1, y = 0.3$; (b) a coax, centered on the z axis, with $a = 0.3, b = 0.5, c = 0.6, I = 2.5$ A in the $\mathbf{a}_z$ direction in the center conductor; (c) three current sheets, $2.7\mathbf{a}_x$ A/m at $y = 0.1, -1.4\mathbf{a}_x$ A/m at $y = 0.15$, and $-1.3\mathbf{a}_x$ A/m at $y = 0.25$.

Ans. $1.989\mathbf{a}_x - 1.989\mathbf{a}_y$ A/m; $-0.884\mathbf{a}_x$ A/m; $1.300\mathbf{a}_z$ A/m

7.3 CURL

We completed our study of Gauss's law by applying it to a differential volume element and were led to the concept of divergence. We now apply Ampère's circuital law to the perimeter of a differential surface element and discuss the third and last of the special derivatives of vector analysis, the curl. Our objective is to obtain the point form of Ampère's circuital law.

Again we choose rectangular coordinates, and an incremental closed path of sides Δx and Δy is selected (Figure 7.13). We assume that some current, as yet unspecified, produces a reference value for $\mathbf{H}$ at the *center* of this small rectangle,

$$\mathbf{H}_0 = H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y + H_{z0}\mathbf{a}_z$$

The closed line integral of $\mathbf{H}$ about this path is then approximately the sum of the four values of $\mathbf{H} \cdot \Delta\mathbf{L}$ on each side. We choose the direction of traverse as 1-2-3-4-1, which corresponds to a current in the $\mathbf{a}_z$ direction, and the first contribution is therefore

$$(\mathbf{H} \cdot \Delta\mathbf{L})_{1-2} = H_{y,1-2}\Delta y$$

The value of H_y on *this section* of the path may be given in terms of the reference value H_{y0} at the center of the rectangle, the rate of change of H_y with x , and the

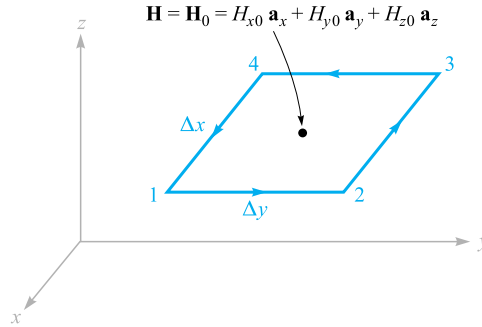


Figure 7.13 An incremental closed path in rectangular coordinates is selected for the application of Ampère's circuital law to determine the spatial rate of change of $\mathbf{H}$.

distance $\Delta x/2$ from the center to the midpoint of side 1–2:

$$H_{y,1-2} \doteq H_{y0} + \frac{\partial H_y}{\partial x} \left(\frac{1}{2} \Delta x \right)$$

Thus

$$(\mathbf{H} \cdot \Delta \mathbf{L})_{1-2} \doteq \left(H_{y0} + \frac{1}{2} \frac{\partial H_y}{\partial x} \Delta x \right) \Delta y$$

Along the next section of the path we have

$$(\mathbf{H} \cdot \Delta \mathbf{L})_{2-3} \doteq H_{x,2-3}(-\Delta x) \doteq - \left(H_{x0} + \frac{1}{2} \frac{\partial H_x}{\partial y} \Delta y \right) \Delta x$$

Continuing for the remaining two segments and adding the results,

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y$$

By Ampère's circuital law, this result must be equal to the current enclosed by the path, or the current crossing any surface bounded by the path. If we assume a general current density $\mathbf{J}$, the enclosed current is then $\Delta I \doteq J_z \Delta x \Delta y$, and

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y \doteq J_z \Delta x \Delta y$$

or

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

As we cause the closed path to shrink, the preceding expression becomes more nearly exact, and in the limit we have the equality

$$\lim_{\Delta x, \Delta y \rightarrow 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \quad (18)$$

After beginning with Ampère's circuital law equating the closed line integral of $\mathbf{H}$ to the current enclosed, we have now arrived at a relationship involving the closed line integral of $\mathbf{H}$ *per unit area* enclosed and the current *per unit area* enclosed, or current density. We performed a similar analysis in passing from the integral form of Gauss's law, involving flux through a closed surface and charge enclosed, to the point form, relating flux through a closed surface *per unit volume* enclosed and charge *per unit volume* enclosed, or volume charge density. In each case a limit is necessary to produce an equality.

If we choose closed paths that are oriented perpendicularly to each of the remaining two coordinate axes, analogous processes lead to expressions for the x and y components of the current density,

$$\lim_{\Delta y, \Delta z \rightarrow 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \quad (19)$$

and

$$\lim_{\Delta z, \Delta x \rightarrow 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \quad (20)$$

Comparing (18)–(20), we see that a component of the current density is given by the limit of the quotient of the closed line integral of $\mathbf{H}$ about a small path in a plane normal to that component and of the area enclosed as the path shrinks to zero. This limit has its counterpart in other fields of science and long ago received the name of *curl*. The curl of any vector is a vector, and any component of the curl is given by the limit of the quotient of the closed line integral of the vector about a small path in a plane normal to that component desired and the area enclosed, as the path shrinks to zero. It should be noted that this definition of curl does not refer specifically to a particular coordinate system. The mathematical form of the definition is

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \rightarrow 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N} \quad (21)$$

where ΔS_N is the planar area enclosed by the closed line integral. The N subscript indicates that the component of the curl is that component which is *normal* to the surface enclosed by the closed path. It may represent any component in any coordinate system.

In rectangular coordinates, the definition (21) shows that the x , y , and z components of the curl $\mathbf{H}$ are given by (18)–(20), and therefore

$$\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{a}_z \quad (22)$$

This result may be written in the form of a determinant,

$$\text{curl } \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \quad (23)$$

and may also be written in terms of the vector operator,

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} \quad (24)$$

Equation (22) is the result of applying the definition (21) to the rectangular coordinate system. We obtained the z component of this expression by evaluating Ampère's circuital law about an incremental path of sides Δx and Δy , and we could have obtained the other two components just as easily by choosing the appropriate paths. Equation (23) is a neat method of storing the rectangular coordinate expression for curl; the form is symmetrical and easily remembered. Equation (24) is even more concise and leads to (22) upon applying the definitions of the cross product and vector operator.

The expressions for curl $\mathbf{H}$ in cylindrical and spherical coordinates are derived in Appendix A by applying the definition (21). Although they may be written in determinant form, as explained there, the determinants do not have one row of unit vectors on top and one row of components on the bottom, and they are not easily memorized. For this reason, the curl expansions in cylindrical and spherical coordinates that follow here and appear inside the back cover are usually referred to whenever necessary.

$$\begin{aligned} \nabla \times \mathbf{H} = & \left(\frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right) \mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) \mathbf{a}_\phi \\ & + \left(\frac{1}{\rho} \frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial H_\rho}{\partial \phi} \right) \mathbf{a}_z \quad (\text{cylindrical}) \end{aligned} \quad (25)$$

$$\begin{aligned} \nabla \times \mathbf{H} = & \frac{1}{r \sin \theta} \left(\frac{\partial(H_\phi \sin \theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi} \right) \mathbf{a}_r + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial(r H_\phi)}{\partial r} \right) \mathbf{a}_\theta \\ & + \frac{1}{r} \left(\frac{\partial(r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right) \mathbf{a}_\phi \quad (\text{spherical}) \end{aligned} \quad (26)$$

Although we have described curl as a line integral per unit area, this does not provide everyone with a satisfactory physical picture of the nature of the curl operation, for the closed line integral itself requires physical interpretation. This integral was first met in the electrostatic field, where we saw that $\oint \mathbf{E} \cdot d\mathbf{L} = 0$. Inasmuch as the integral was zero, we did not belabor the physical picture. More recently we have discussed the closed line integral of $\mathbf{H}$, $\oint \mathbf{H} \cdot d\mathbf{L} = I$. Either of these closed line integrals is also known by the name of *circulation*, a term borrowed from the field of fluid dynamics.

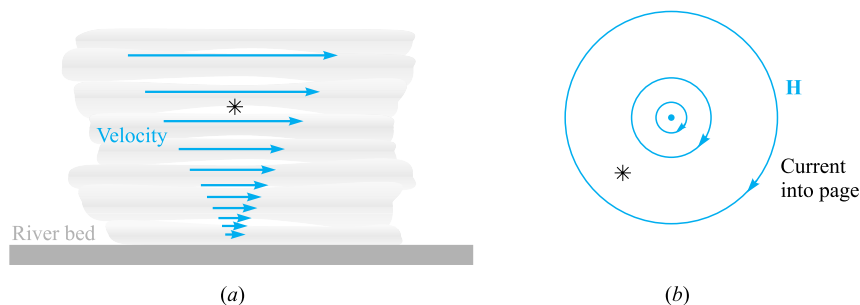


Figure 7.14 (a) The curl meter shows a component of the curl of the water velocity into the page. (b) The curl of the magnetic field intensity about an infinitely long filament is shown.

The circulation of $\mathbf{H}$, or $\oint \mathbf{H} \cdot d\mathbf{L}$, is obtained by multiplying the component of $\mathbf{H}$ parallel to the specified closed path at each point along it by the differential path length and summing the results as the differential lengths approach zero and as their number becomes infinite. We do not require a vanishingly small path. Ampère's circuital law tells us that if $\mathbf{H}$ does possess circulation about a given path, then current passes through this path. In electrostatics we see that the circulation of $\mathbf{E}$ is zero about every path, a direct consequence of the fact that zero work is required to carry a charge around a closed path.

We may describe curl as *circulation per unit area*. The closed path is vanishingly small, and curl is defined at a point. The curl of $\mathbf{E}$ must be zero, for the circulation is zero. The curl of $\mathbf{H}$ is not zero, however; the circulation of $\mathbf{H}$ per unit area is the current density by Ampère's circuital law [or (18), (19), and (20)].

Skilling⁵ suggests the use of a very small paddle wheel as a “curl meter.” Our vector quantity, then, must be thought of as capable of applying a force to each blade of the paddle wheel, the force being proportional to the component of the field normal to the surface of that blade. To test a field for curl, we dip our paddle wheel into the field, with the axis of the paddle wheel lined up with the direction of the component of curl desired, and note the action of the field on the paddle. No rotation means no curl; larger angular velocities mean greater values of the curl; a reversal in the direction of spin means a reversal in the sign of the curl. To find the direction of the vector curl and not merely to establish the presence of any particular component, we should place our paddle wheel in the field and hunt around for the orientation which produces the greatest torque. The direction of the curl is then along the axis of the paddle wheel, as given by the right-hand rule.

As an example, consider the flow of water in a river. Figure 7.14a shows the longitudinal section of a wide river taken at the middle of the river. The water velocity is zero at the bottom and increases linearly as the surface is approached. A paddle wheel placed in the position shown, with its axis perpendicular to the paper, will turn in a clockwise direction, showing the presence of a component of curl in the direction

⁵ See the References at the end of the chapter.