Lect. 6

RLC circuit Natural and Step Response.

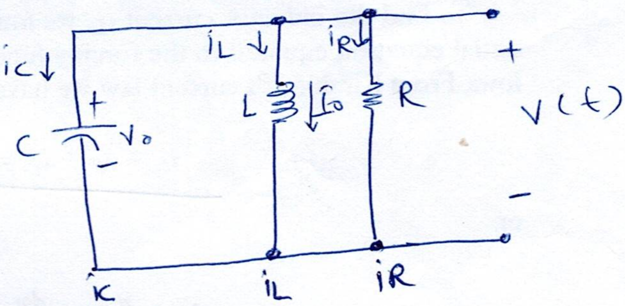
- RLC
  - parallel
    - NR
    - SR
  - Serise
    - NR
    - SR

overdamped
underdamped
critically damped

+ Introduction to the Natural Response of a parallel RLC cct.

$$i_R + i_L + i_C = 0$$

$$\frac{V}{R} + \left(\frac{1}{L}\int_0^t V\,dt + I_0\right) + C\frac{dV}{dt} = 0$$



نشتق للتخلص من التكامل

$$\therefore \frac{1}{R}\frac{dV}{dt} + \frac{1}{L}\cdot V + C\frac{d^2V}{dt^2} = 0 \Big] \div C$$

$$\Rightarrow \boxed{\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{V}{LC} = 0}$$

$$\longrightarrow V = V_0 e^{-t/\tau}$$

sub $V = A\cdot e^{st}$

$$\therefore \frac{dV}{dt} = A\cdot s\cdot e^{st}$$

$$\frac{d^2V}{dt^2} = A s^2 e^{st}$$

$$As^2e^{st} + \frac{1}{RC} Ase^{st} + \frac{1}{LC} A.e^{st} = 0$$

$$Ae^{st}\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right) = 0$$

$$s_1 = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$, \quad \alpha = \frac{1}{2RC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$s_2 = \frac{-1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$\therefore s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$\alpha \rightarrow$ Neper freqency

$\omega_0 \rightarrow$ Resonant radian frequency

s, ω_0, α [rad/s] unit
radian

For both s_1, s_2

$$v_1 = A_1 e^{s_1 t}$$

$$v_2 = A_2 e^{s_2 t}$$

$$v = v_1 + v_2$$

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t} \longrightarrow \text{over damped}$$

three stats

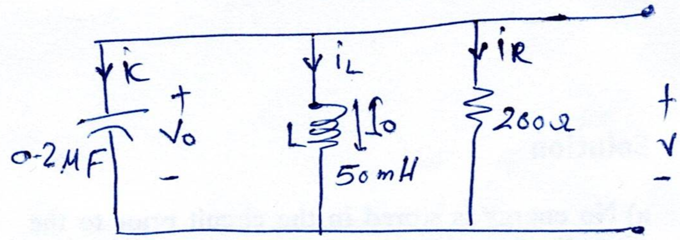
$\alpha^2 > \omega_0^2 \longrightarrow$ overdamp

$\alpha^2 < \omega_0^2 \longrightarrow$ underdamp

$\alpha^2 = \omega_0^2 \longrightarrow$ criticallydamp

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Example:



a) Find the roots (S_1 + S_2)

b) type of response

c) Repeat a & b for R = 312.5 Ω

d) what value of R causes the Response to be critically damped?

Sol:

$$S_1 = -\alpha + \sqrt{\alpha^2 - w_0^2}$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2*200*0.2*10^{-6}} = 1.25*10^4 \text{ rad/s}, \quad \alpha^2 = 1.5625*10^8 \text{ rad}^2/s^2$$

$$w_0^2 = \frac{1}{LC} = \frac{1}{0.2*10^{-6}*50*10^{-3}} = 10^8 \text{ rad}^2/s^2$$

$$\therefore S_1 = -1.25*10^4 + \sqrt{(1.25*10^4)^2 - 10^8}$$

$$\boxed{S_1 = -5000 \text{ rad/s}}$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - w_0^2}$$

$$= -1.25*10^4 - \sqrt{(1.25*10^4)^2 - 10^8}$$

$$\boxed{S_2 = -20000 \text{ rad/s}}$$

(b) $\alpha^2 > w_0^2 \longrightarrow$ over damped

(c) R = 312.5

$$\alpha = \frac{-1}{2RC} = \frac{-1}{2*312.5*0.2*10^{-6}} = 8000 \text{ rad/s}, \quad \alpha^2 = 0.64*10^8 \text{ rad}^2/s^2$$

$$w_0^2 = 10^8 \text{ rad}^2/s^2$$

$$\therefore S_1 = -\alpha + \sqrt{\alpha^2 - w_0^2}$$

$$s_1 = -8000 + \sqrt{0.64 * 10^8 - 10^8} = -8000 + \sqrt{-64 * 10^6}$$

$$\boxed{s_1 = -8000 + j6000 \text{ rad/s}}$$

$$\boxed{s_2 = -8000 - j6000 \text{ rad/s}}$$

in this case

$\alpha^2 < w_0^2 \longrightarrow$ under dumped

(d) critically means $\longrightarrow w_0^2 = \alpha^2$

$$\left(\frac{1}{2RC}\right)^2 = \underbrace{\frac{1}{LC}}_{10^8}$$

$$\therefore \left(\frac{1}{2RC}\right)^2 = 10^8 \Big] \sqrt{\ } \Rightarrow \frac{1}{2RC} = 10^4 \Rightarrow R = \frac{1}{2 * 10^4 * \underbrace{0.2 * 10^{-6}}_{C}}$$

$R = 250\ \Omega$

* The Forms of NR of Parallel RLC cct. (5)

⇒ overdamped voltage Response.

$$V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \rightarrow \frac{dv(t)}{dt} = A_1 S_1 e^{s_1 t} + A_2 S_2 e^{s_2 t}$$

$$V(0) = A_1 + A_2$$

$$\frac{dv(0)}{dt} = S_1 A_1 + S_2 A_2$$

$$\because \; i_C = C \frac{dv(0)}{dt} \Rightarrow \boxed{\frac{i_C(0)}{C} = \frac{dv(0)}{dt} = S_1 A_1 + S_2 A_2}$$

$\therefore$

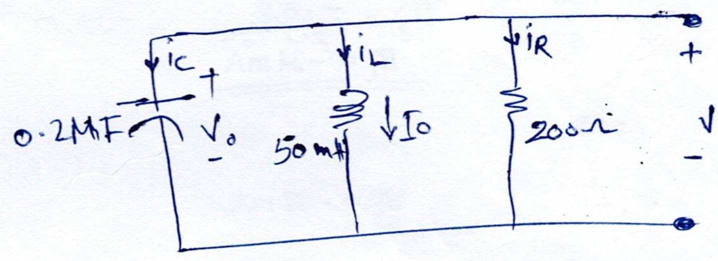
خطوات الحل :-

1. S_1 , S_2
2. $V(0) , \frac{dv(0)}{dt}$
3. A_1 , A_2
4. Sub A_1, A_2, S_1 and S_2 in $\rightarrow \boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$

Example

$V(0) = 12$ V. , $R = 200\ \Omega$

$i_L(0) = 30$ mA



a- initial current for The each branch $i_L(0)$, $i_R(0)$, $i_C(0)$

b- $\frac{dv(0)}{dt}$

c) $v(t)$

~~d- v(t) ...~~

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Solution :

(a) $i_L(0) = 30\ mA$

$i_R(0) = \frac{V_0}{R} = \frac{12}{200} = 60\ mA$

$i_c(0) = -i_L(0) - i_R(0)$ → $i_c + i_R + i_L = 0 \Rightarrow i_c = -i_R - i_L$

$= -30 - 60 = -90\ mA$

(b) $\frac{dv(0)}{dt} = \frac{i_c(0)}{c} = \frac{-90 \times 10^{-3}}{0.2 \times 10^{-6}} = -450\ KV/s$

(c) $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$s_1 = -\alpha + \sqrt{\alpha^2 - w_0^2}$

$\alpha = \frac{1}{2RC} = 1.25 \times 10^4\ rad/s$, $\alpha^2 = 1.5625 \times 10^8\ rad^2/s^2$

$w_0^2 = \frac{1}{LC} = \frac{1}{0.2 \times 10^{-6} \times 50 \times 10^{-3}} = 10^8$

$w_0^2 = 10^8\ rad^2/s^2$

$\boxed{\therefore s_1 = -5000\ rad/s}$

$s_2 = -\alpha - \sqrt{\alpha^2 - w_0^2}$

$\therefore \boxed{s_2 = -20000\ rad/s}$

$\therefore \alpha^2 > w_0^2 \Longrightarrow$ overdamped

$V(0) = A_1 + A_2$

$12 = A_1 + A_2$ ------ ①

$\frac{dV(0)}{dt} = A_1 s_1 + A_2 s_2$

$-450 = -5000 A_1 - 20000 A_2$ ------ ②

$A_1 = -14\ v.$

$A_2 = 26\ v.$

sub in $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$v(t) = \left(-14 e^{-5000t} + 26 e^{-20000t}\right) V \quad t \geq 0$

* The underdamped voltage Respons

$-\alpha \pm \sqrt{\alpha^2 - w_0^2}$ هذه الحالة

سبب السالب تحت الجذر

$w_0^2 > \alpha^2$

$$S_1 = -\alpha + \sqrt{-(w_0^2 - \alpha^2)}$$
$$= -\alpha + j\sqrt{w_0^2 - \alpha^2}$$

$$\boxed{\begin{aligned} S_1 &= -\alpha + jwd \\ S_2 &= -\alpha - jwd \end{aligned}}$$

where $\boxed{wd = \sqrt{w_0^2 - \alpha^2}}$

wd: damped radian frequency.

$$\boxed{v(t) = B_1 e^{-\alpha t} \cos w_d t + B_2 e^{-\alpha t} \sin w_d t}$$ underdamped NR

$$V(0) = B_1 (1)(1) + B_2 (1)(0)$$

$$\boxed{V(0) = B_1}$$

$$\frac{dv(0)}{dt} = \frac{ic(0)}{c} = \boxed{-\alpha B_1 + w_d B_2}$$

خطوات الحل :-

① find α, w_d, B_1, B_2

② $V(0)$, $\frac{dv(0)}{dt}$

③ $v(t)$, $t \geq 0$

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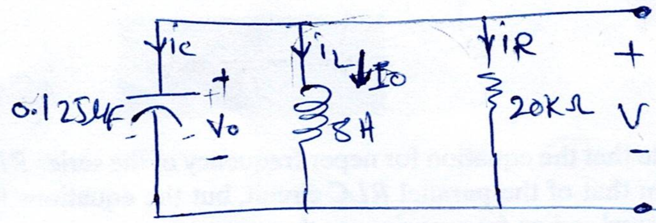
Example : $V_0 = 0$, $I_0 = -12.25\ mA$

a) Find roots s_1, s_2

b) v , $\frac{dv}{dt}$ at $t = 0^+$

c) $v(t)$, $t \geq 0$



SoL

a) $\alpha = \frac{1}{2RC} = \frac{10^6}{2(20)10^3(0.125)} = 200\ rad/s$

$w_0 = \frac{1}{\sqrt{LC}} = \sqrt{\frac{1}{(8)(10^{-3})(0.125*10^{-6})}} = 10^3\ rad/s$

$\therefore \alpha^2 < w_0^2 \rightarrow$ underdamped

$s_1 = -\alpha + jwd$, $s_2 = -\alpha - jwd$

$wd = \sqrt{w_0^2 - \alpha^2} = \sqrt{(10^3)^2 - (200)^2} = 100\sqrt{96}\ rad/s = 979.80\ rad/s$

$\therefore s_1 = -200 + j\,979.80\ rad/s$

$s_2 = -200 - j\,979.80\ rad/s$

b) $v(0) = V_0 = 0$, $i_R = \frac{V_0}{R} = 0 \Rightarrow i_c = -i_R - i_L$

$i_c = -i_L$

$i_c(0) = -(-12.25) = 12.25\ mA$

$\therefore \frac{dv(0)}{dt} = \frac{i_c(0)}{c} = \frac{12.25*10^{-3}}{0.125*10^{-6}} = 98000\ V/s$

c) $\boxed{v(t) = B_1 e^{-\alpha t}\cos wdt + B_2 e^{-\alpha t}\sin w_d t}$

$v(0) = B_1 = 0$, $B_2 = ?$

$\frac{dv(0)}{dt} = -\alpha B_1 + wdB_2$

$98000 = 0 + (979.8)*B_2$

$B_2 \cong 100\ V.$

$\boxed{v(t) = 0 + 100e^{-200t}\sin(979.8)t}\ V$

The critically damped voltage Response

$\alpha^2 = \omega_0^2$

$$\left.\begin{aligned} s_1 &= -\alpha + \sqrt{\cancel{\omega_0^2 - \alpha^2}} \\ s_2 &= -\alpha - \sqrt{\cancel{\omega_0^2 - \alpha^2}} \end{aligned}\right\} \longrightarrow s_1 = s_2 = -\alpha$$

(→ 0)

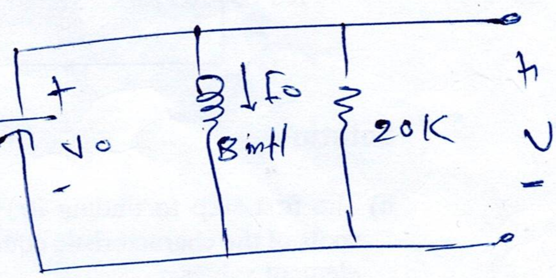
$$\boxed{v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}} \longrightarrow v(0) = 0 + D_2$$

$$\frac{dv(t)}{dt} = e^{-\alpha t}(D_1) + (D_1 t + D_2)(-\alpha e^{-\alpha t}) \Rightarrow \frac{dv(0)}{dt} = D_1 - \alpha D_2$$

$$\boxed{\frac{dv(0)}{dt} = D_1 - \alpha D_2} = \frac{i_C(0)}{C} = \frac{-(I_0 + V_0/R)}{C}$$

$$\boxed{v(0) = D_2}$$

Example , $V_0 = 0$, $i_0 = -12.25$ mA



(a) find R that result critically damped, (b) v(t)

Sol

$\omega_0^2 = 10^6$ → from Ex ①

in critically $\alpha^2 = \omega_0^2$

$\alpha = 10^3$, $\alpha = \dfrac{1}{2RC}$

$$10^3 = \frac{1}{2(0.125 \times 10^{-6})R} \Rightarrow R = 4000\,\Omega = 4\,k\Omega$$

من هنا
أنطيهم
باعتبار $\alpha = \omega = 10^3$ مع تغير قيمة $1/4R$ $\underline{4K}$

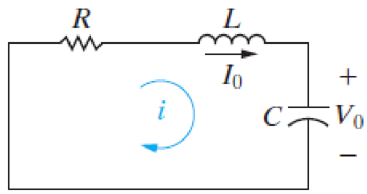
$v(t) \Rightarrow$? $\quad v(0) = 0 = D_2$

$$\frac{dv(0)}{dt} = D_1 - \alpha D_2 = \frac{i_C(0)}{C} = \frac{12.25 \times 10^{-3}}{0.125 \times 10^{-6}} = 98000 \text{ V/s}$$

$\therefore\ 98000 = D_1 - 0 \Rightarrow D_1 = 98000$ V/s

$$\boxed{v(t) = 98000\, t e^{-1000t}} \text{ V} \quad t \geq 0$$

2. Natural Response of Series RLC.



$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i d\tau + V_0 = 0.$$

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0,$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0.$$

$$i = Ae^{st}$$

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

or

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

Since $i = Ae^{st}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

A more compact way of expressing the roots is

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

$$\boxed{\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}}$$

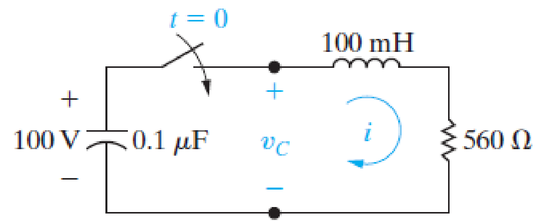
1. If $\alpha > \omega_0$, we have the *overdamped* case.
2. If $\alpha = \omega_0$, we have the *critically damped* case.
3. If $\alpha < \omega_0$, we have the *underdamped* case.

Current natural response forms in series RLC circuits

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(overdamped)},$$
$$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t \quad \text{(underdamped)},$$
$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \quad \text{(critically damped)}.$$

Example

The 0.1 μF capacitor in the circuit shown in Fig. 8.16 is charged to 100 V. At $t = 0$ the capacitor is discharged through a series combination of a 100 mH inductor and a 560 Ω resistor.



a) Find $i(t)$ for $t \geq 0$.

b) Find $v_C(t)$ for $t \geq 0$.

Solution

a) The first step to finding $i(t)$ is to calculate the roots of the characteristic equation. For the given element values,

$$\omega_0^2 = \frac{1}{LC} = \frac{(10^3)(10^6)}{(100)(0.1)} = 10^8,$$

$$\alpha = \frac{R}{2L} = \frac{560}{2(100)} \times 10^3 = 2800 \text{ rad/s}.$$

Next, we compare ω_0^2 to α^2 and note that $\omega_0^2 > \alpha^2$, because

$$\alpha^2 = 7.84 \times 10^6 = 0.0784 \times 10^8.$$

At this point, we know that the response is underdamped and that the solution for $i(t)$ is of the form

$$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t,$$

where $\alpha = 2800$ rad/s and $\omega_d = 9600$ rad/s. The numerical values of B_1 and B_2 come from the initial conditions. The inductor current is zero before the switch has been closed, and hence it is zero immediately after. Therefore

$$i(0) = 0 = B_1.$$

To find B_2, we evaluate $di(0^+)/dt$. From the circuit, we note that, because $i(0) = 0$ immediately after the switch has been closed, there will be no voltage drop across the resistor. Thus the initial voltage on the capacitor appears across the terminals of the inductor, which leads to the expression,

$$Vl = L\frac{di}{dt} = Vo$$

$$\frac{di}{dt} = \frac{Vo}{L} = \frac{100}{100} * 10^3 = 1000 \ \ A/s$$

$$\frac{di(0)}{dt} = w_d \, B_2 - \alpha B_1$$

$$1000 = (9600)(B_2) - 0$$

$$B_2 = \frac{1000}{9600} = 0.1042 \quad A$$

$$i(t) = B_2 \, e^{-\alpha t} \sin w_d \, t \quad = 0.1042 \, e^{-2800t} \sin 9600 \, t \qquad A$$

b) To find $v_C(t)$, we can use either of the following relationships:

$$v_C = -\frac{1}{C}\int_0^t i \, d\tau + 100 \text{ or}$$

$$v_C = iR + L\frac{di}{dt}.$$

Whichever expression is used (the second is recommended), the result is

$$v_C(t) = (100 \cos 9600t + 29.17 \sin 9600t)e^{-2800t} \text{ V}, \quad t \geq 0.$$