

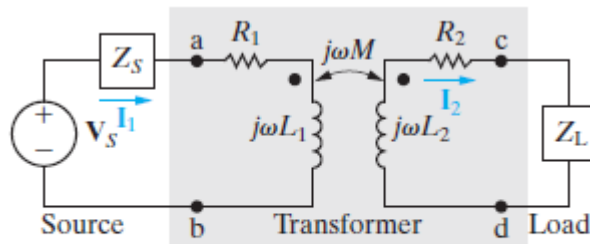
The Transformer

A transformer is a device that is based on magnetic coupling. Transformers are used in both communication and power circuits. In communication circuits, the transformer is used to match impedances and eliminate dc signals from portions of the system. In power circuits, transformers are used to establish ac voltage levels that facilitate the transmission, distribution, and consumption of electrical power. A knowledge of the sinusoidal steady-state

behavior of the transformer is required in the analysis of both communication and power systems. In this section, we will discuss the sinusoidal steady-state behavior of the **linear transformer**, which is found primarily in communication circuits. And then , we will deal with the **ideal transformer**, which is used to model the ferromagnetic transformer found in power systems.

Before starting we make a useful observation. When analyzing circuits containing mutual inductance use the meshor loop-current method for writing circuit equations. The node-voltage method is cumbersome to use when mutual inductance is involved. This is because the currents in the various coils cannot be written by inspection as functions of the node voltages.

The Analysis of a Linear Transformer Circuit



▲ The frequency domain circuit model for a transformer used to connect a load to a source.

A simple **transformer** is formed when two coils are wound on a single core to ensure magnetic coupling. Figure . shows the frequency-domain circuit model of a system that uses a transformer to connect a load to a source. In discussing this circuit, we refer to the transformer winding connected to the source as the **primary winding** and the winding connected to the load as the **secondary winding**. Based on this terminology, the transformer circuit parameters are

R_1 = the resistance of the primary winding,

R_2 = the resistance of the secondary winding,

L_1 = the self-inductance of the primary winding,

L_2 = the self-inductance of the secondary winding,

M = the mutual inductance.

The internal voltage of the sinusoidal source is $\mathbf{V}_s$, and the internal impedance of the source is Z_s . The impedance Z_L represents the load connected to the secondary winding of the transformer. The phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$ represent the primary and secondary currents of the transformer, respectively.

Analysis of the circuit in Figure consists of finding $\mathbf{I}_1$ and $\mathbf{I}_2$ as functions of the circuit parameters $\mathbf{V}_s$, Z_s , R_1 , L_1 , L_2 , R_2 , M , Z_L , and ω . We are also interested in finding the impedance seen looking into the transformer from the terminals a,b. To find $\mathbf{I}_1$ and $\mathbf{I}_2$, we first write the two mesh-current equations that describe the circuit:

$$\mathbf{V}_s = (Z_s + R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2,$$

$$0 = -j\omega M\mathbf{I}_1 + (R_2 + j\omega L_2 + Z_L)\mathbf{I}_2.$$

To facilitate the algebraic manipulation of Eqs. we let:

$$Z_{11} = Z_s + R_1 + j\omega L_1,$$

$$Z_{22} = R_2 + j\omega L_2 + Z_L,$$

where Z_{11} is the total self-impedance of the mesh containing the primary winding of the transformer, and Z_{22} is the total self-impedance of the mesh containing the secondary winding. Based on the notation introduced in Equations above , the solutions for $\mathbf{I}_1$ and $\mathbf{I}_2$ are :

$$\mathbf{I}_1 = \frac{Z_{22}}{Z_{11}Z_{22} + \omega^2 M^2} \mathbf{V}_s,$$

$$\mathbf{I}_2 = \frac{j\omega M}{Z_{11}Z_{22} + \omega^2 M^2} \mathbf{V}_s = \frac{j\omega M}{Z_{22}} \mathbf{I}_1.$$

To the internal source voltage V_s , the impedance appears as V_s/I_1 , or

$$\frac{V_s}{I_1} = Z_{\text{int}} = \frac{Z_{11}Z_{22} + \omega^2 M^2}{Z_{22}} = Z_{11} + \frac{\omega^2 M^2}{Z_{22}}.$$

The impedance at the terminals of the source is $Z_{\text{int}} - Z_s$, so

$$Z_{\text{ab}} = Z_{11} + \frac{\omega^2 M^2}{Z_{22}} - Z_s = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{(R_2 + j\omega L_2 + Z_L)}.$$

Note that the impedance Z_{ab} is independent of the magnetic polarity of the transformer. The reason is that the mutual inductance appears in Eq. Z_{ab} as a squared quantity. This impedance is of particular interest because it shows how the transformer affects the impedance of the load as seen from the source. Without the transformer, the load would be connected directly to the source, and the source would see a load impedance of Z_L ; with the transformer, the load is connected to the source through the transformer, and the source sees a load impedance that is a modified version of Z_L , as seen in the third term of Equation of Z_{ab}

Reflected Impedance

The third term in Eq. 9.64 is called the **reflected impedance** (Z_r), because it is the equivalent impedance of the secondary coil and load impedance transmitted, or reflected, to the primary side of the transformer. Note that the reflected impedance is due solely to the existence of mutual inductance; that is, if the two coils are decoupled, M becomes zero, Z_r becomes zero, and Z_{ab} reduces to the self-impedance of the primary coil.

To consider reflected impedance in more detail, we first express the load impedance in rectangular form:

$$Z_L = R_L + jX_L,$$

where the load reactance X_L carries its own algebraic sign. In other words, X_L is a positive number if the load is inductive and a negative number if the load is capacitive. We now use Eq. 9.64 to write the reflected impedance in rectangular form:

$$\begin{aligned} Z_r &= \frac{\omega^2 M^2}{R_2 + R_L + j(\omega L_2 + X_L)} \\ &= \frac{\omega^2 M^2 [(R_2 + R_L) - j(\omega L_2 + X_L)]}{(R_2 + R_L)^2 + (\omega L_2 + X_L)^2} \\ &= \frac{\omega^2 M^2}{|Z_{22}|^2} [(R_2 + R_L) - j(\omega L_2 + X_L)]. \end{aligned}$$

The derivation of Eq. (10.10) takes advantage of the fact that, when Z_L is written in rectangular form, the self-impedance of the mesh containing the secondary winding is

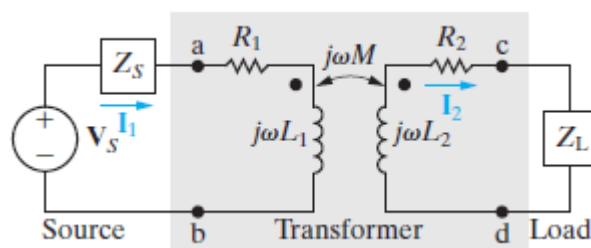
$$Z_{22} = R_2 + R_L + j(\omega L_2 + X_L).$$

Now observe from Eq. (10.10) that the self-impedance of the secondary circuit is reflected into the primary circuit by a scaling factor of $(\omega M/|Z_{22}|)^2$, and that the sign of the reactive component $(\omega L_2 + X_L)$ is reversed. Thus the linear transformer reflects the conjugate of the self-impedance of the secondary circuit (Z_{22}^*) into the primary winding by a scalar multiplier.

Example

The parameters of a certain linear transformer are $R_1 = 200 \, \Omega$, $R_2 = 100 \, \Omega$, $L_1 = 9 \, \text{H}$, $L_2 = 4 \, \text{H}$, and $k = 0.5$. The transformer couples an impedance consisting of an $800 \, \Omega$ resistor in series with a $1 \, \mu\text{F}$ capacitor to a sinusoidal voltage source. The $300 \, \text{V}$ (rms) source has an internal impedance of $500 + j100 \, \Omega$ and a frequency of $400 \, \text{rad/s}$.

- Construct a frequency-domain equivalent circuit of the system.
- Calculate the self-impedance of the primary circuit.
- Calculate the self-impedance of the secondary circuit.
- Calculate the impedance reflected into the primary winding.
- Calculate the scaling factor for the reflected impedance.
- Calculate the impedance seen looking into the primary terminals of the transformer.
- Calculate the Thévenin equivalent with respect to the terminals c,d.



Solution

- a) Figure shows the frequency-domain equivalent circuit. Note that the internal voltage of the source serves as the reference phasor, and that V_1 and V_2 represent the terminal voltages of the transformer. In constructing the circuit in Figure , we made the following calculations:

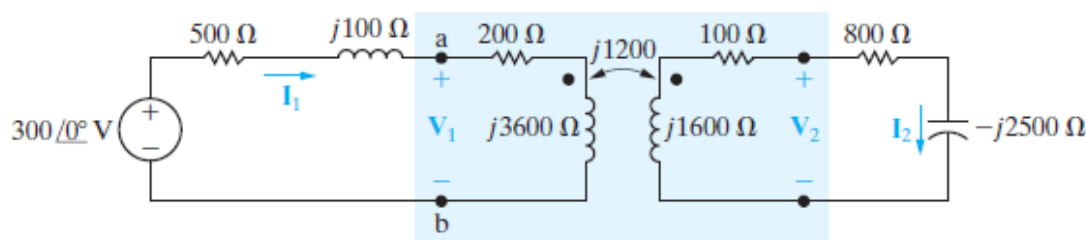
$$j\omega L_1 = j(400)(9) = j3600 \Omega,$$

$$j\omega L_2 = j(400)(4) = j1600 \Omega,$$

$$M = 0.5\sqrt{(9)(4)} = 3 \text{ H},$$

$$j\omega M = j(400)(3) = j1200 \Omega,$$

$$\frac{1}{j\omega C} = \frac{10^6}{j400} = -j2500 \Omega.$$



▲ The frequency-domain equivalent circuit for Example

- b) The self-impedance of the primary circuit is

$$Z_{11} = 500 + j100 + 200 + j3600 = 700 + j3700 \Omega.$$

- c) The self-impedance of the secondary circuit is

$$Z_{22} = 100 + j1600 + 800 - j2500 = 900 - j900 \Omega.$$

- d) The impedance reflected into the primary winding is

$$\begin{aligned} Z_r &= \left(\frac{1200}{|900 - j900|} \right)^2 (900 + j900) \\ &= \frac{8}{9} (900 + j900) = 800 + j800 \Omega. \end{aligned}$$

- e) The scaling factor by which Z_{22}^* is reflected is 8/9.

- f) The impedance seen looking into the primary terminals of the transformer is the impedance of the primary winding plus the reflected impedance; thus

$$Z_{ab} = 200 + j3600 + 800 + j800 = 1000 + j4400 \Omega.$$

- g) The Thévenin voltage will equal the open circuit value of V_{cd} . The open circuit value of V_{cd} will equal $j1200$ times the open circuit value of I_1 . The open circuit value of I_1 is

$$\begin{aligned} I_1 &= \frac{300 \angle 0^\circ}{700 + j3700} \\ &= 79.67 \angle -79.29^\circ \text{ mA.} \end{aligned}$$

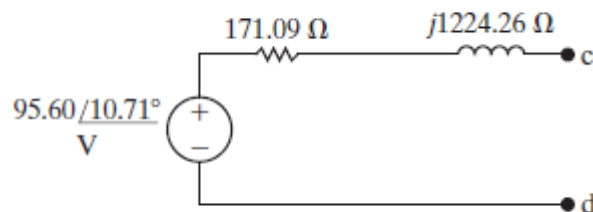
Therefore

$$\begin{aligned} V_{Th} &= j1200(79.67 \angle -79.29^\circ) \times 10^{-3} \\ &= 95.60 \angle 10.71^\circ \text{ V.} \end{aligned}$$

The Thévenin impedance will be equal to the impedance of the secondary winding plus the impedance reflected from the primary when the voltage source is replaced by a short-circuit. Thus

$$\begin{aligned} Z_{Th} &= 100 + j1600 + \left(\frac{1200}{|700 + j3700|} \right)^2 (700 - j3700) \\ &= 171.09 + j1224.26 \Omega. \end{aligned}$$

The Thévenin equivalent is shown in Fig. 9.40.



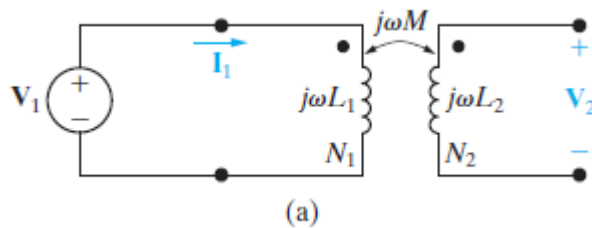
▲ The Thévenin equivalent circuit for Example

The Ideal Transformer

An **ideal transformer** consists of two magnetically coupled coils having N_1 and N_2 turns, respectively, and exhibiting these three properties:

1. The coefficient of coupling is unity ($k = 1$).
2. The self-inductance of each coil is infinite ($L_1 = L_2 = \infty$).
3. The coil losses, due to parasitic resistance, are negligible.

Determining the Voltage and Current Ratios



Note in Fig. (a) that the voltage at the terminals of the open-circuit coil is entirely the result of the current in coil 1; therefore

$$\mathbf{V}_2 = j\omega M \mathbf{I}_1.$$

The current in coil 1 is

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{j\omega L_1}.$$

From Eqs.

$$\mathbf{V}_2 = \frac{M}{L_1} \mathbf{V}_1.$$

For unity coupling, the mutual inductance equals $\sqrt{L_1 L_2}$, so Eq. becomes

$$\mathbf{V}_2 = \sqrt{\frac{L_2}{L_1}} \mathbf{V}_1.$$

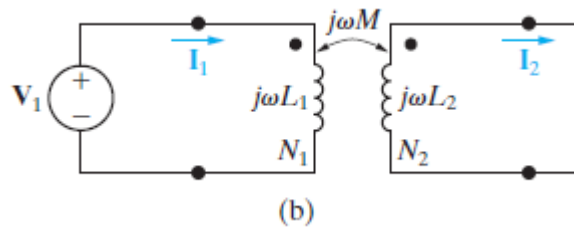
For unity coupling, the flux linking coil 1 is the same as the flux linking coil 2, so we need only one permeance to describe the self-inductance of each coil. Thus Eq. becomes

$$\mathbf{V}_2 = \sqrt{\frac{N_2^2 \mathcal{P}}{N_1^2 \mathcal{P}}} \mathbf{V}_1 = \frac{N_2}{N_1} \mathbf{V}_1$$

or

Voltage relationship for an ideal transformer ►

$$\frac{V_1}{N_1} = \frac{V_2}{N_2}.$$



Summing the voltages around the shorted coil of Fig. (b) yields

$$0 = -j\omega M I_1 + j\omega L_2 I_2,$$

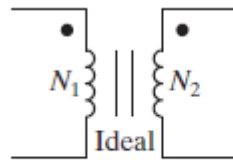
from which, for $k = 1$,

$$\frac{I_1}{I_2} = \frac{L_2}{M} = \frac{L_2}{\sqrt{L_1 L_2}} = \sqrt{\frac{L_2}{L_1}} = \frac{N_2}{N_1}.$$

Equation is equivalent to

$$I_1 N_1 = I_2 N_2.$$

◄ **Current relationship for an ideal transformer**



▲ The graphic symbol for an ideal **transformer**

Figure shows the graphic symbol for an ideal transformer. The vertical lines in the symbol represent the layers of magnetic material from which ferromagnetic cores are often made. Thus, the symbol reminds us that coils wound on a ferromagnetic core behave very much like an ideal transformer.

There are several reasons for this. The ferromagnetic material creates a space with high permeance. Thus most of the magnetic flux is trapped inside the core material, establishing tight magnetic coupling between coils that share the same core. High permeance also means high self-inductance, because $L = N^2 \mathcal{P}$. Finally, ferromagnetically coupled coils efficiently transfer power from one coil to the other. Efficiencies in excess of 95% are common, so neglecting losses is not a crippling approximation for many applications.

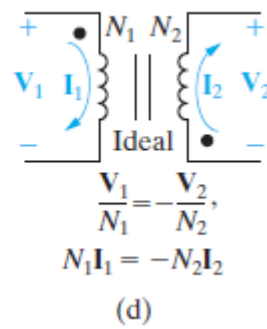
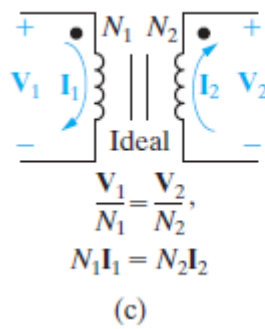
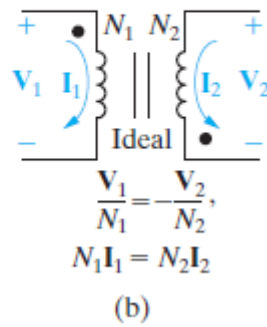
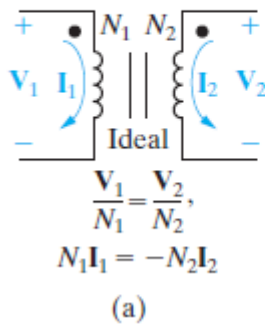
Determining the Polarity of the Voltage and Current Ratios

If the coil voltages V_1 and V_2 are both positive or negative at the dot-marked terminal, use a plus sign in Eq. Otherwise, use a negative sign.

If the coil currents I_1 and I_2 are both directed into or out of the dot-marked terminal, use a minus sign in Eq. Otherwise, use a plus sign.

Dot convention for ideal transformers

The four circuits shown in Fig. illustrate these rules



The ratio of the turns on the two windings is an important parameter of the ideal transformer. The turns ratio is defined as either N_1/N_2 or N_2/N_1 ; both ratios appear in various writings. In this text, we use a to denote the ratio N_2/N_1 , or

$$a = \frac{N_2}{N_1}.$$

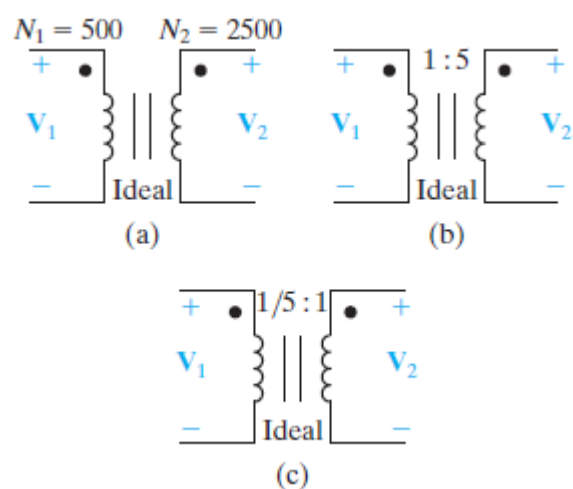
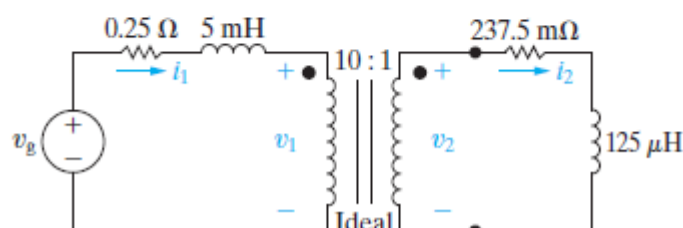


Figure 1 shows three ways to represent the turns ratio of an ideal transformer. Figure 1(a) shows the number of turns in each coil explicitly. Figure 1(b) shows that the ratio N_2/N_1 is 5 to 1, and Fig. 1(c) shows that the ratio N_2/N_1 is 1 to $\frac{1}{5}$.

Example Analyzing an Ideal Transformer Circuit

The load impedance connected to the secondary winding of the ideal transformer in Fig. consists of a $237.5\text{ m}\Omega$ resistor in series with a $125\text{ }\mu\text{H}$ inductor.

If the sinusoidal voltage source (v_g) is generating the voltage $2500 \cos 400t\text{ V}$, find the steady-state expressions for: (a) i_1 ; (b) v_1 ; (c) i_2 ; and (d) v_2 .



Solution

- a) We begin by constructing the phasor domain equivalent circuit. The voltage source becomes $2500\angle 0^\circ\text{ V}$; the 5 mH inductor converts to an impedance of $j2\text{ }\Omega$; and the $125\text{ }\mu\text{H}$ inductor converts to an impedance of $j0.05\text{ }\Omega$. The phasor domain equivalent circuit is shown in Fig. 9.46.

It follows directly from Fig. 9.46 that

$$2500\angle 0^\circ = (0.25 + j2)\mathbf{I}_1 + \mathbf{V}_1,$$

and

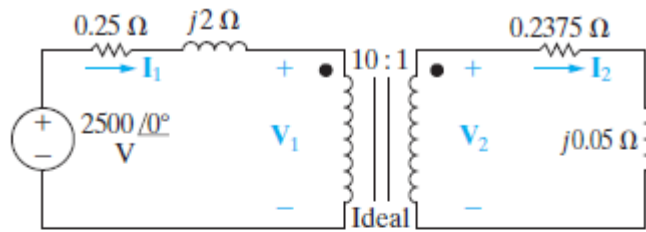
$$\mathbf{V}_1 = 10\mathbf{V}_2 = 10[(0.2375 + j0.05)\mathbf{I}_2].$$

Because

$$\mathbf{I}_2 = 10\mathbf{I}_1$$

we have

$$\begin{aligned}\mathbf{V}_1 &= 10(0.2375 + j0.05)10\mathbf{I}_1 \\ &= (23.75 + j5)\mathbf{I}_1.\end{aligned}$$



Therefore

$$2500 \angle 0^\circ = (24 + j7)\mathbf{I}_1,$$

or

$$\mathbf{I}_1 = 100 \angle -16.26^\circ \text{ A.}$$

Thus the steady-state expression for i_1 is

$$i_1 = 100 \cos(400t - 16.26^\circ) \text{ A.}$$

$$\begin{aligned} \text{b) } \mathbf{V}_1 &= 2500 \angle 0^\circ - (100 \angle -16.26^\circ)(0.25 + j2) \\ &= 2500 - 80 - j185 \\ &= 2420 - j185 = 2427.06 \angle -4.37^\circ \text{ V.} \end{aligned}$$

Hence

$$v_1 = 2427.06 \cos(400t - 4.37^\circ) \text{ V.}$$

$$\text{c) } \mathbf{I}_2 = 10\mathbf{I}_1 = 1000 \angle -16.26^\circ \text{ A.}$$

Therefore

$$i_2 = 1000 \cos(400t - 16.26^\circ) \text{ A.}$$

$$\text{d) } \mathbf{V}_2 = 0.1\mathbf{V}_1 = 242.71 \angle -4.37^\circ \text{ V,}$$

giving

$$v_2 = 242.71 \cos(400t - 4.37^\circ) \text{ V.}$$

The Use of an Ideal Transformer for Impedance Matching

Ideal transformers can also be used to raise or lower the impedance level of a load. The circuit shown in Fig. 9.90 illustrates this. The impedance seen by the practical voltage source (V_s in series with Z_s) is V_1/I_1 . The voltage and current at the terminals of the load impedance (V_2 and I_2) are related to V_1 and I_1 by the transformer turns ratio; thus

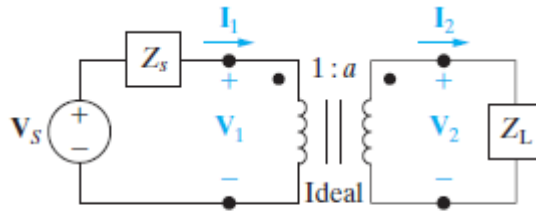


Fig. 9.90 Using an ideal transformer to couple a load to a source.

$$V_1 = \frac{V_2}{a},$$

and

$$I_1 = aI_2.$$

Therefore the impedance seen by the practical source is

$$Z_{IN} = \frac{V_1}{I_1} = \frac{1}{a^2} \frac{V_2}{I_2},$$

but the ratio V_2/I_2 is the load impedance Z_L , so Eq. 9.90 becomes

$$Z_{IN} = \frac{1}{a^2} Z_L.$$

Thus, the ideal transformer's secondary coil reflects the load impedance back to the primary coil, with the scaling factor $1/a^2$.

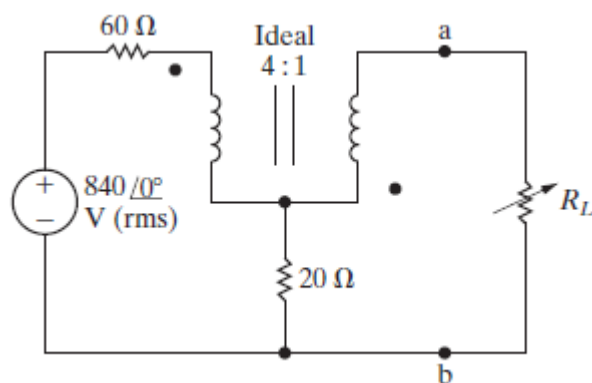
Note that the ideal transformer changes the magnitude of Z_L but does not affect its phase angle. Whether Z_{IN} is greater or less than Z_L depends on the turns ratio a .

The ideal transformer—or its practical counterpart, the ferromagnetic core transformer—can be used to match the magnitude of Z_L to the magnitude of Z_s .

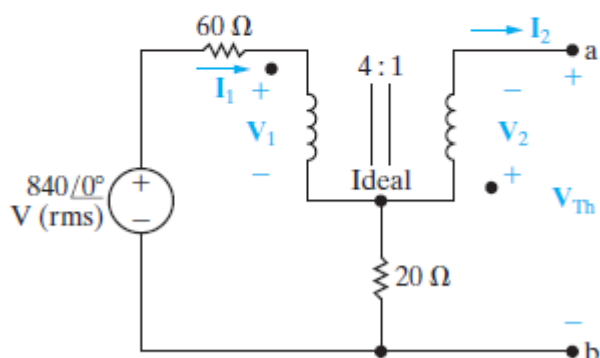
Example**Finding Maximum Power Transfer in a Circuit with an Ideal Transformer**

The variable resistor in the circuit in Figure is adjusted until maximum average power is delivered to R_L .

- What is the value of R_L in ohms?
- What is the maximum average power (in watts) delivered to R_L ?



- We first find the Thévenin equivalent with respect to the terminals of R_L . The circuit for determining the open circuit voltage is shown in Fig. . The variables V_1 , V_2 , I_1 , and I_2 have been added to expedite the discussion.



▲ The circuit used to find the Thévenin voltage.

First we note the ideal transformer imposes the following constraints on the variables $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{I}_1$, and $\mathbf{I}_2$:

$$\mathbf{V}_2 = \frac{1}{4} \mathbf{V}_1, \quad \mathbf{I}_1 = -\frac{1}{4} \mathbf{I}_2.$$

The open circuit value of $\mathbf{I}_2$ is zero, hence $\mathbf{I}_1$ is zero. It follows that

$$\mathbf{V}_1 = 840 \angle 0^\circ \text{ V}, \quad \mathbf{V}_2 = 210 \angle 0^\circ \text{ V}.$$

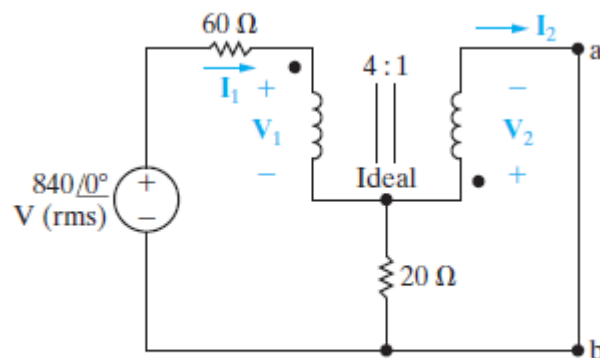
From Fig. we note that $\mathbf{V}_{\text{Th}}$ is the negative of $\mathbf{V}_2$, hence

$$\mathbf{V}_{\text{Th}} = -210 \angle 0^\circ \text{ V}.$$

The circuit shown in Fig. 10.26 is used to determine the short circuit current. Viewing $\mathbf{I}_1$ and $\mathbf{I}_2$ as mesh currents, the two mesh equations are

$$840 \angle 0^\circ = 80\mathbf{I}_1 - 20\mathbf{I}_2 + \mathbf{V}_1,$$

$$0 = 20\mathbf{I}_2 - 20\mathbf{I}_1 + \mathbf{V}_2.$$



▲ The circuit used to calculate the short circuit current.

When these two mesh current equations are combined with the constraint equations we get

$$840 \angle 0^\circ = -40\mathbf{I}_2 + \mathbf{V}_1,$$

$$0 = 25\mathbf{I}_2 + \frac{\mathbf{V}_1}{4}.$$

Solving for the short circuit value of $\mathbf{I}_2$ yields

$$\mathbf{I}_2 = -6 \text{ A.}$$

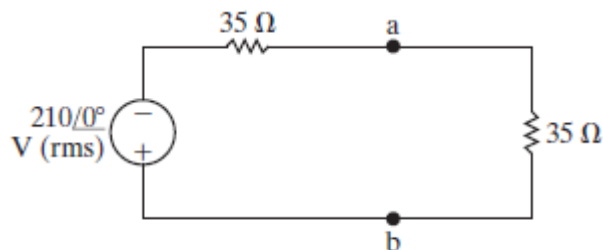
Therefore the Thévenin resistance is

$$R_{\text{Th}} = \frac{-210}{-6} = 35 \, \Omega.$$

Maximum power will be delivered to R_L when R_L equals $35 \, \Omega$.

- b) The maximum power delivered to R_L is most easily determined using the Thévenin equivalent. From the circuit shown in Fig. we have

$$P_{\text{max}} = \left(\frac{-210}{70} \right)^2 (35) = 315 \text{ W.}$$



▲ The Thévenin equivalent loaded for maximum power transfer.