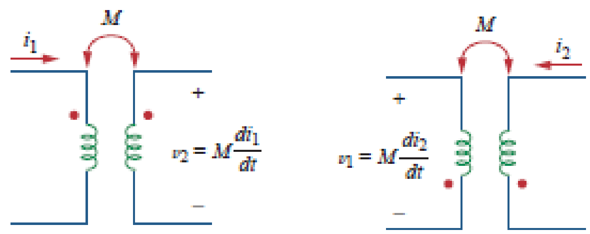
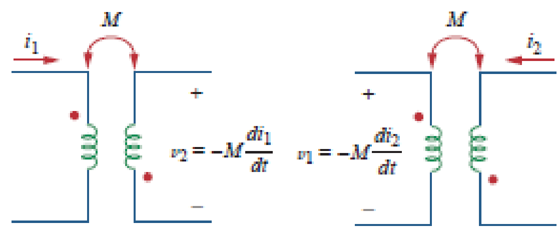
Mutual Inductance

Mutual inductance is the ability of one inductor to induce a voltage across a neighboring inductor, measured in henrys (H).

$$L = L_1 + L_2 + 2M \qquad \text{(Series-aiding connection)}$$



$$L = L_1 + L_2 - 2M \qquad \text{(Series-opposing connection)}$$



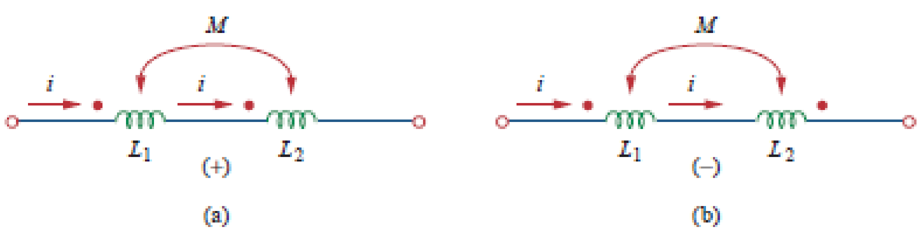


Figure
Dot convention for coils in series; the sign indicates the polarity of the mutual voltage:
(a) series-aiding connection, (b) series-opposing connection

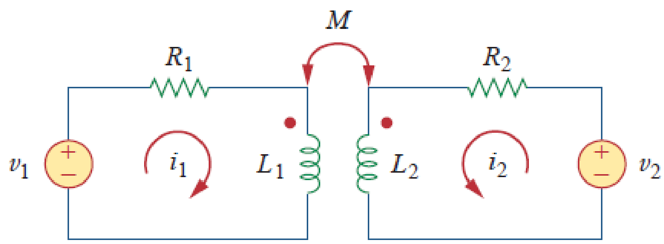


Figure
Time-domain analysis of a circuit containing coupled coils.

KVL to coil 1 gives

$$v_1 = i_1 R_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

For coil 2, KVL gives

$$v_2 = i_2 R_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

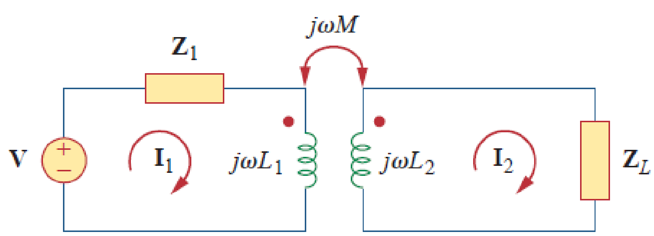


Figure
Frequency-domain analysis of a circuit containing coupled coils.

in the frequency domain. Applying KVL to coil 1, we get

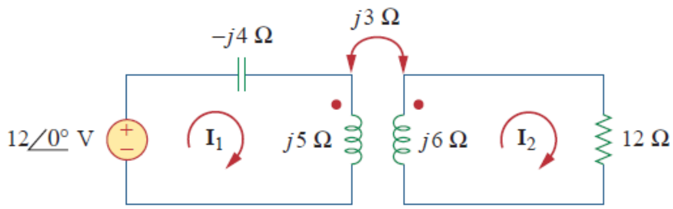
$$\mathbf{V} = (\mathbf{Z}_1 + j\omega L_1)\mathbf{I}_1 - j\omega M \mathbf{I}_2$$

For coil 2, KVL yields

$$0 = -j\omega M \mathbf{I}_1 + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2$$

Calculate the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit of Fig. 13.9.

Example



Solution:

For coil 1, KVL gives

$$-12 + (-j4 + j5)\mathbf{I}_1 - j3\mathbf{I}_2 = 0$$

or

$$j\mathbf{I}_1 - j3\mathbf{I}_2 = 12$$

For coil 2, KVL gives.

$$-j3\mathbf{I}_1 + (12 + j6)\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_1 = \frac{(12 + j6)\mathbf{I}_2}{j3} = (2 - j4)\mathbf{I}_2$$

Substituting this in

$$(j2 + 4 - j3)\mathbf{I}_2 = (4 - j)\mathbf{I}_2 = 12$$

or

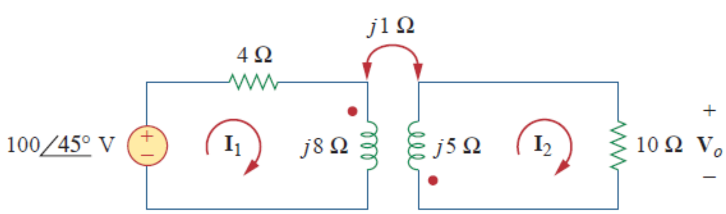
$$\mathbf{I}_2 = \frac{12}{4 - j} = 2.91\angle 14.04° \text{ A}$$

From Eqs. (13.1.2) and (13.1.3),

$$\begin{aligned}\mathbf{I}_1 = (2 - j4)\mathbf{I}_2 &= (4.472\angle -63.43°)(2.91\angle 14.04°)\\ &= 13.01\angle -49.39° \text{ A}\end{aligned}$$

Practice Problem

Determine the voltage $\mathbf{V}_o$ in the circuit



Energy in a Coupled Circuit

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2$$

Coefficient of coupling k

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

The coupling coefficient k is a measure of the magnetic coupling between two coils; 0 ≤ k ≤ 1

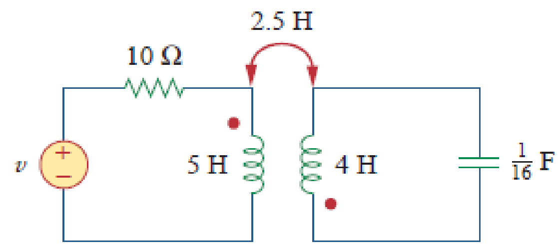
$k = 1$ *perfectly coupled*

$k < 0.5,$ *loosely coupled*

$k > 0.5,$ *tightly coupled*

Example

Consider the circuit in Fig. Determine the coupling coefficient. Calculate the energy stored in the coupled inductors at time $t = 1$ s if $v = 60\cos(4t + 30^\circ)$ V.



Solution:

The coupling coefficient is

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

indicating that the inductors are tightly coupled. To find the energy stored, we need to calculate the current. To find the current, we need to obtain the frequency-domain equivalent of the circuit.

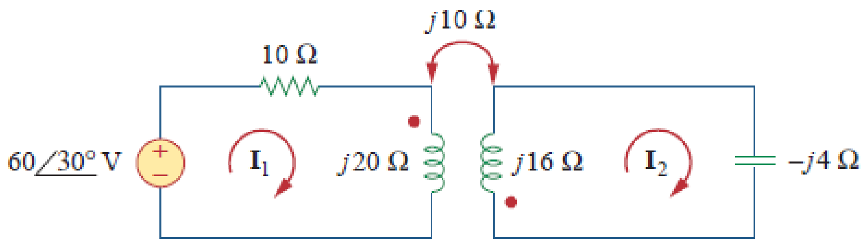
$$60\cos(4t + 30^\circ) \quad \Rightarrow \quad 60\underline{/30^\circ}, \quad \omega = 4 \text{ rad/s}$$

$$5 \text{ H} \quad \Rightarrow \quad j\omega L_1 = j20\ \Omega$$

$$2.5 \text{ H} \quad \Rightarrow \quad j\omega M = j10\ \Omega$$

$$4 \text{ H} \quad \Rightarrow \quad j\omega L_2 = j16\ \Omega$$

$$\frac{1}{16}\text{F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j4\ \Omega$$



We now apply mesh analysis. For mesh 1

$$(10 + j20)\mathbf{I}_1 + j10\mathbf{I}_2 = 60\underline{/30^\circ}$$

For mesh 2,

$$j10\mathbf{I}_1 + (j16 - j4)\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_1 = -1.2\mathbf{I}_2$$

$$\mathbf{I}_2(-12 - j14) = 60\underline{/30^\circ} \quad \Rightarrow \quad \mathbf{I}_2 = 3.254\underline{/\qquad} \text{ A}$$

and

$$\mathbf{I}_1 = -1.2\mathbf{I}_2 = 3.905\underline{/-19.4^\circ} \text{ A}$$

In the time-domain,

$$i_1 = 3.905\cos(4t - 19.4^\circ), \qquad i_2 = 3.254\cos(4t + 160.6^\circ)$$

At time $t = 1$ s, $4t = 4$ rad $= 229.2^\circ$, and

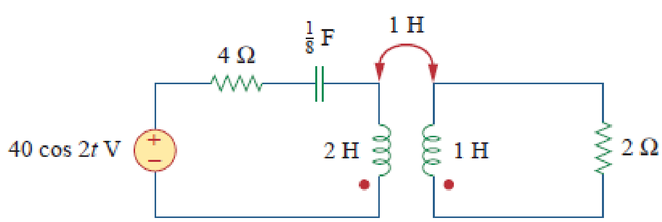
$$i_1 = 3.905\cos(229.2^\circ - 19.4^\circ) = -3.389 \text{ A}$$
$$i_2 = 3.254\cos(229.2^\circ + 160.6^\circ) = 2.824 \text{ A}$$

The total energy stored in the coupled inductors is

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2$$
$$= \frac{1}{2}(5)(-3.389)^2 + \frac{1}{2}(4)(2.824)^2 + 2.5(-3.389)(2.824) = 20.73 \text{ J}$$

Practice Problem

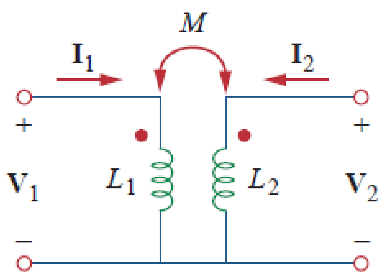
For the circuit in Fig. determine the coupling coefficient and the energy stored in the coupled inductors at $t = 1.5$ s.



Linear Transformers

A transformer is a magnetic device that takes advantage of the phenomenon of mutual inductance. A transformer is generally a four-terminal device comprising two (or more) magnetically coupled coils. It is used in changing the current, voltage, or impedance level in a circuit.

A linear (or loosely coupled) transformer has its coils wound on a magnetically linear material. It can be replaced by an equivalent T or π network for the purposes of analysis.



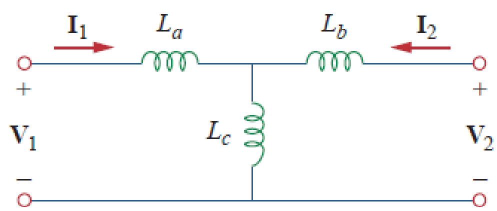
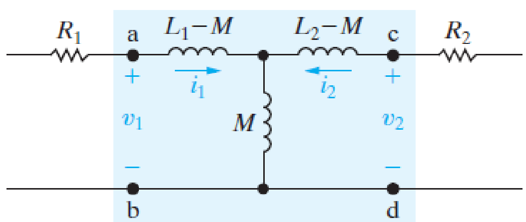


Figure
An equivalent T circuit.

the equivalent circuit of a linear transformer

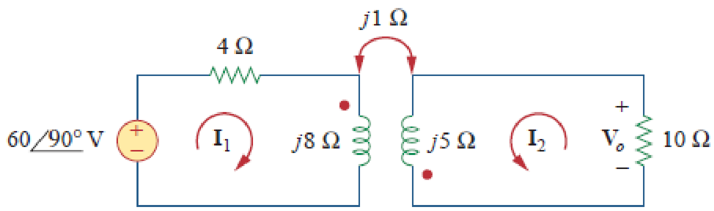


$$\boxed{L_a = L_1 - M, \qquad L_b = L_2 - M, \qquad L_c = M}$$

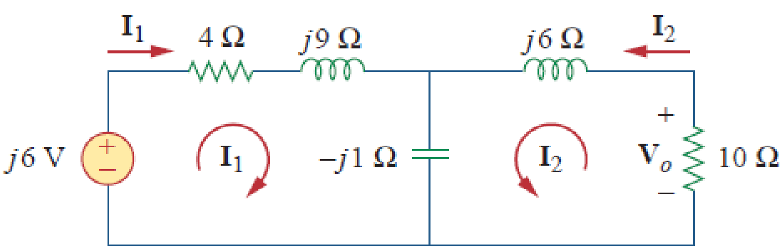
$$x_L = jwL = j2\pi f c$$

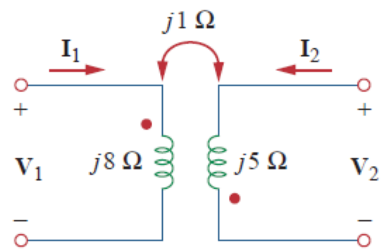
$$x_C = \frac{1}{jwc} = -\frac{j}{2\pi f c}$$

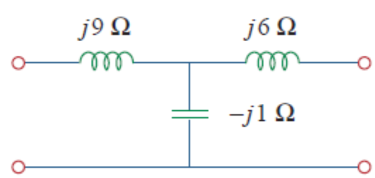
Example Solve for **I**1, **I**2, and **V***o* using the T-equivalent circuit for the linear transormer.



$$L_a = L_1 - (-M) = 8 + 1 = 9\text{ H}$$
$$L_b = L_2 - (-M) = 5 + 1 = 6\text{ H}, \qquad L_c = -M = -1\text{ H}$$







$$j6 = \mathbf{I}_1(4 + j9 - j1) + \mathbf{I}_2(-j1)$$

$$0 = \mathbf{I}_1(-j1) + \mathbf{I}_2(10 + j6 - j1)$$

$$\mathbf{I}_1 = \frac{(10 + j5)}{j}\mathbf{I}_2 = (5 - j10)\mathbf{I}_2$$

$$j6 = (4 + j8)(5 - j10)\mathbf{I}_2 - j\mathbf{I}_2 = (100 - j)\mathbf{I}_2 \simeq 100\mathbf{I}_2$$

Since 100 is very large compared with 1, the imaginary part of $(100 - j)$ can be ignored so that $100 - j \simeq 100$. Hence,

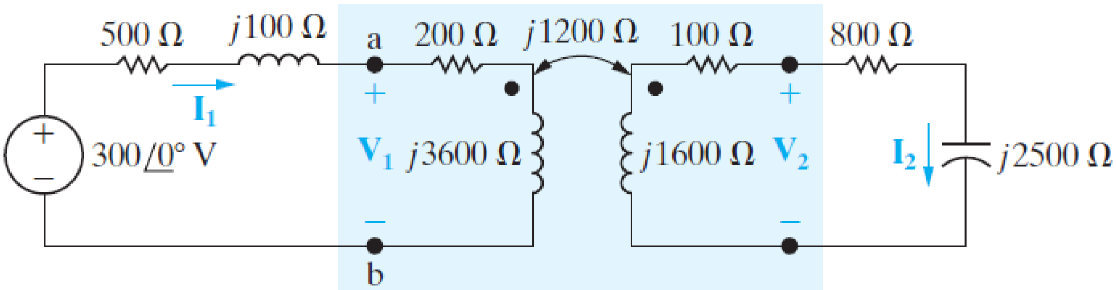
$$\mathbf{I}_2 = \frac{j6}{100} = j0.06 = 0.06\underline{/90°}\text{ A}$$

$$\mathbf{I}_1 = (5 - j10)j0.06 = 0.6 + j0.3\text{ A}$$

and

$$\mathbf{V}_o = -10\mathbf{I}_2 = -j0.6 = 0.6\underline{/-90°}\text{ V}$$

Q/

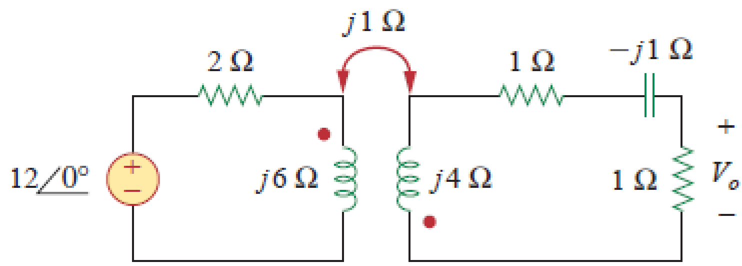


a) Use the T-equivalent circuit for the magnetically coupled coils shown in Fig. C.6 to find the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$. The source frequency is 400 rad/s.

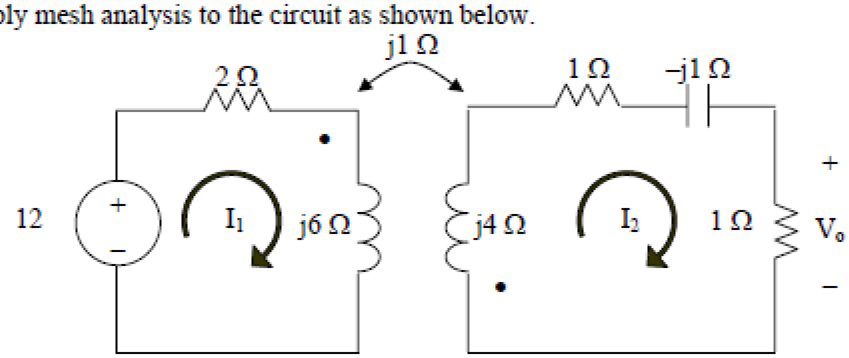
b) Repeat (a), but with the polarity dot on the secondary winding moved to the lower terminal.

Mutual inductance examples

Q1/ For the circuit in Fig., find *Vo*.



We apply mesh analysis to the circuit as shown below.



For mesh 1,

$$12 = I_1(2 + j6) + jI_2 \qquad (1)$$

For mesh 2,

$$0 = jI_1 + (2 - j1 + j4)I_2$$

or

$$0 = jI_1 + (2 + j3)I_2 \qquad (2)$$

In matrix form,

$$\begin{bmatrix} 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+j6 & j \\ j & 2+j3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$I_2 = -0.4381 + j0.3164$$

$$V_o = I_2 x1 = \underline{\mathbf{540.5\angle 144.16^\circ \ mV}}.$$

$$\Delta = \begin{vmatrix} 2+j6 & j \\ j & 2+j3 \end{vmatrix} = \text{4+j2+j6-18+1= -13+j18}$$

$$\Delta 1 = \begin{matrix} 12 & j \\ 0 & 2+j3 \end{matrix} = \text{24+j36}$$

$$\Delta 2 = \begin{vmatrix} 2+j6 & 12 \\ j & 0 \end{vmatrix} = 0 - j12 = -j12$$

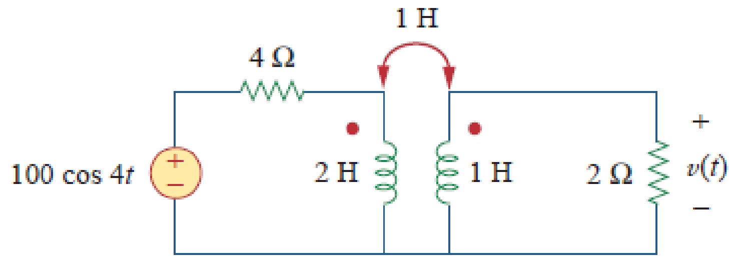
$$I_1 = \frac{\Delta_1}{\Delta} = \frac{24+j36}{-13+j18} = \frac{43.26\angle 56.31}{22.2\angle -54.16} = 1.94\angle 110.47$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{-j12}{-13+j18} = \frac{(-j12)*(-13-j18)}{(-13+j18)*(-13-j18)} = \frac{j156-216}{493} =$$

$$= -0.438 + j0.316$$

الحل اعلاه هو بطريقة المصفوفة يمكنك اتباعها.

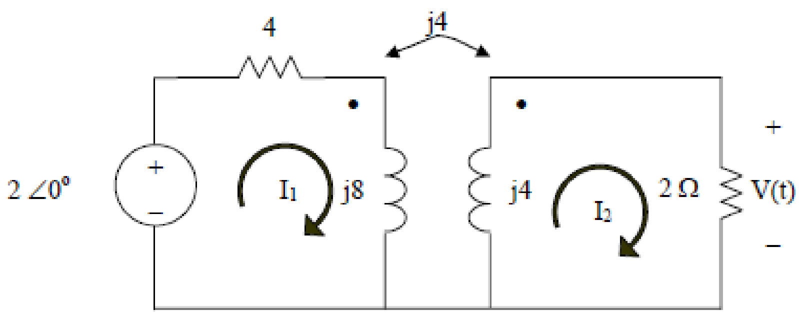
Q2/Find *v*(*t*) for the circuit in Fig.



$2H \longrightarrow j\omega L = j4x2 = j8$

$1H \longrightarrow j\omega L = j4x1 = j4$

Consider the circuit below.



$$2 = (4 + j8)I_1 - j4I_2 \qquad (1)$$
$$0 = -j4I_1 + (2 + j4)I_2 \qquad (2)$$

In matrix form, these equations become

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4+j8 & -j4 \\ -j4 & 2+j4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Solving this leads to

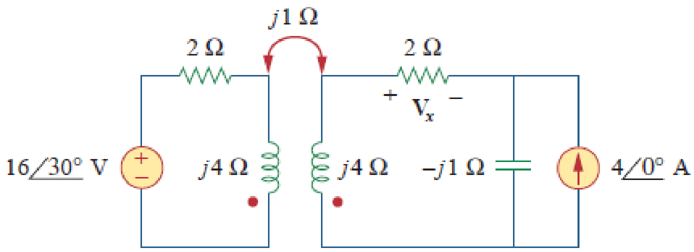
$I_2 = 0.2353 - j0.0588$

$V = 2I_2 = 0.4851 \angle -14.04^\circ$

Thus,

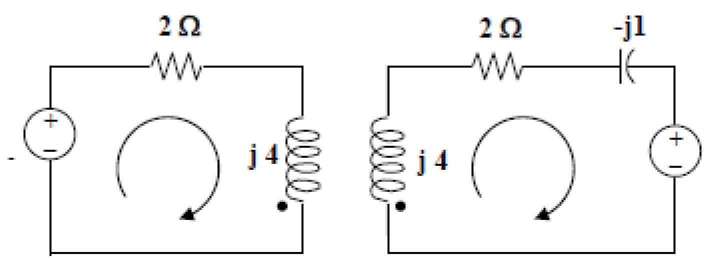
$v(t) = 0.4851\cos(4t - 14.04^\circ)$ V

Q3/Find V_x for the circuit in Fig.



- عن طريق تحويل مصدر التيار الى مصدر فولتية
- افرض اتجاه التيار كما ترغب

Consider the circuit below.



For loop 1,

$$8\angle 30^\circ = (2 + j4)I_1 - jI_2 \qquad (1)$$

For loop 2, $((j4 + 2 - j)I_2 - jI_1 + (-j2) = 0$

or $I_1 = (3 - j2)i_2 - 2 \qquad (2)$

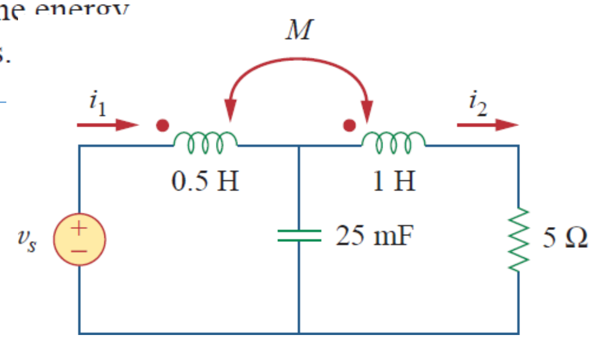
Substituting (2) into (1), $8\angle 30^\circ + (2 + j4)2 = (14 + j7)I_2$

$$I_2 = (10.928 + j12)/(14 + j7) = 1.037\angle 21.12^\circ$$

$$V_x = 2I_2 = \underline{\mathbf{2.074\angle 21.12^\circ}}$$

يمكنك حل المعادلات بطريقة التعويض او بطريقة المصفوفة كما في السؤال الاول

Q4/ If $M = 0.2$ H and $v_s = 120 \cos 10t$ V in the circuit of Fig. , find i_1 and i_2. Calculate the energy stored in the coupled coils at $t = 15$ ms.



$\omega = 10$

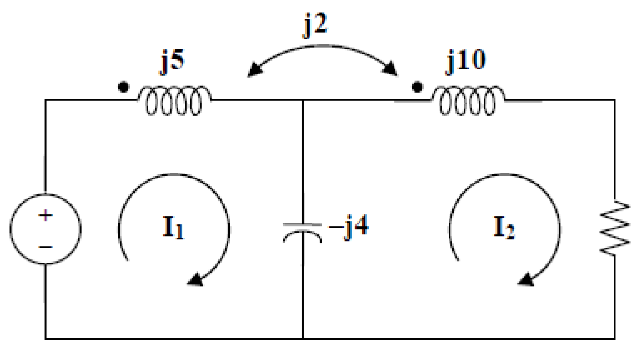
0.5 H converts to $j\omega L_1 = j5$ ohms

1 H converts to $j\omega L_2 = j10$ ohms

0.2 H converts to $j\omega M = j2$ ohms

25 mF converts to $1/(j\omega C) = 1/(10\text{x}25\text{x}10^{-3}) = -j4$ ohms

The frequency-domain equivalent circuit is shown below.



For mesh 1, $12 = (j5 - j4)I_1 + j2I_2 - (-j4)I_2$

$$-j12 = I_1 + 6I_2 \qquad (1)$$

For mesh 2, $0 = (5 + j10)I_2 + j2I_1 - (-j4)I_1$

$$0 = (5 + j10)I_2 + j6I_1 \qquad (2)$$

From (1), $I_1 = -j12 - 6I_2$

Substituting this into (2) produces,

$$I_2 = 72/(-5 + j26) = 2.7194\angle{-100.89°}$$

$$I_1 = -j12 - 6\,I_2 = -j12 - 163.17\angle{-100.89} = 5.068\angle{52.54°}$$

Hence,

$i_1 = \underline{\mathbf{5.068cos(10t + 52.54°)\ A}}$, $i_2 = \underline{\mathbf{2.719cos(10t - 100.89°)\ A}}$.

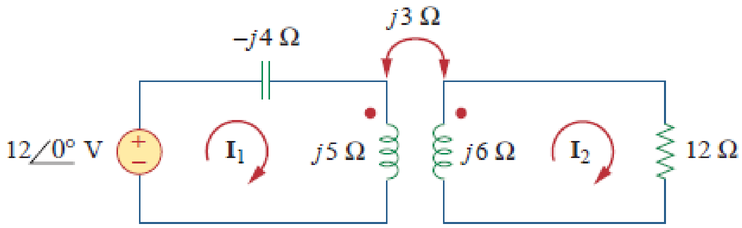
At t = 15 ms, $10t = 10\text{x}15\text{x}10^{-3}\ 0.15 \text{ rad} = 8.59°$

$$i_1 = 5.068\cos(61.13°) = 2.446$$

$$i_2 = 2.719\cos(-92.3°) = -0.1089$$

$$w = 0.5(5)(2.446)^2 + 0.5(1)(-0.1089)^2 - (0.2)(2.446)(-0.1089) = \underline{\mathbf{15.02\ J}}$$

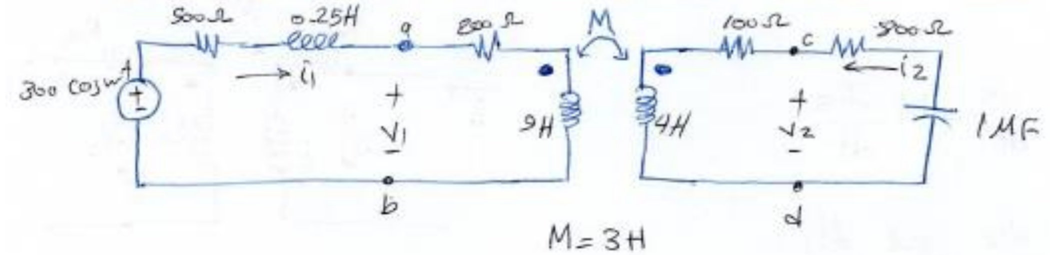
Q5/ Calculate the phasor currents I1 and I2 in the circuit using T-equivalent cct for linear transformer.



Answer: $13\angle 49.4$ A,

$2.91\angle 14.04$ A.

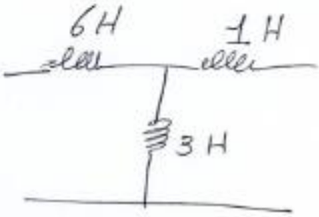
Ex: Use the T-equ. cct. for the magnitally coupled coils shown to find the phaser current (I_1, I_2), $w = 400$ rad/s.



Sol:

T-equ. cct. needs

$$\left.\begin{aligned} L_1 - M &= 9 - 3 = 6\ H \\ L_2 - M &= 4 - 3 = 1\ H \\ M &= 3\ H \end{aligned}\right\}$$



So we have to Find XL, Xc

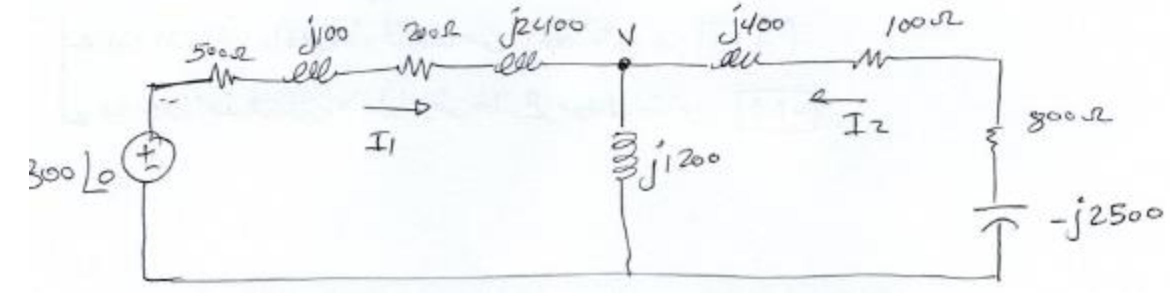
$X_{L1} = jwL_1 = j(400)(6) = j\,2400\ \Omega$

$X_{L2} = jwL_2 = j(400)(1) = j\,400\ \Omega$

$X_{L3} = jwM = j(400)(3) = j\,1200\ \Omega$

$X_{L4} = jw(0.25H) = j(400)(0.25) = j\,100\ \Omega$

$X_C = \dfrac{j}{wC} = j\dfrac{1}{(400)(10^{-6})} = j\,2500\ \Omega$



To Find I_1, I_2 , we will use Nodel method

$$\frac{V-300}{700+j2500}+\frac{V}{j1200}+\frac{V}{900-j2100}=0$$

$$V=136-j8 \quad =136.24\angle -3.37 \quad V.$$

$$\text{then } I_1=\frac{300-(136-j8)}{700+j2500}=63.25\angle -71.57 \text{ mA}$$

$$I_2=\frac{136-j8}{900-j2100}=59.63\angle 63.43 \text{ mA}$$