i. Distinct poles

1- Factorization of denominator
2- Divide LHS and RHS by z
3- Create partial fractions and find constants
4- Find X(z)
5- Find x(n)

$$\frac{\mathbf{X}(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \dots + \frac{A_N}{z-p_N}$$

$$A_i = \lim_{z \to p_i} (z-p_i)\frac{\mathbf{X}(z)}{z}$$

Example : Find $x(n)$ for $\mathrm{X}(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$ for the following ROC's:

$a)$ $|z| > 1$, b) $|z| < 0.5$ and c) $0.5 < |z| < 1$

Solution: $\mathrm{X}(z) = \dfrac{z^2}{z^2-1.5z+0.5} \rightarrow$

$$\frac{\mathrm{X}(z)}{z} = \frac{z}{(z-1)(Z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

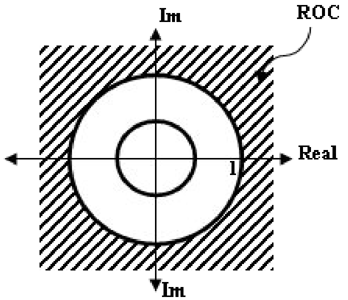
$$A_1 = \lim_{z\to 1}\frac{z}{(z-0.5)} = 2, \quad A_2 = \lim_{Z\to 0.5}\frac{z}{(z-1)} = -1$$

$$\frac{\mathrm{X}(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5},$$

$$\mathbf{X}(z) = \mathbf{2}\frac{z}{z-1} - \frac{z}{z-0.5}$$

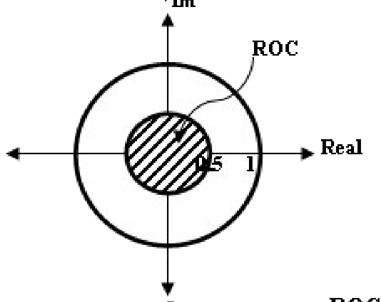
<u>*a. ROC* $|z| > 1$</u>

$$x(n) = 2u(n) - (0.5)^n u(n)$$
$$= \{1, 3/2, 7/4, 15/8, 31/16, \dots\}.$$



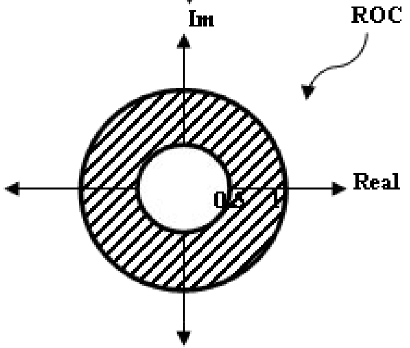
<u>***b. ROC*** $|Z| < 0.5$</u>

$$x(n) = -2u(-n-1) + (0.5)^n u(-n-1)$$



<u>***c. ROC*** $0.5 < |z| < 1$</u>

$$x(n) = -2u(-n-1) - (0.5)^n u(n)$$



ii. Multiple order poles

$$\text{If } \frac{X(z)}{z} = \frac{d_o + d_1 z + d_2 z^2 + \dots + d_{M-1} z^{M-1}}{b_o + b_1 z + b_2 z^2 + \dots + b_M z^M} = \frac{A_{1k}}{(z-z_r)^k} + \frac{A_{1k-1}}{(z-z_r)^{k-1}} + \dots + \frac{A_{11}}{z-z_r} + \sum_{j=k+1}^{M} \frac{A_j}{z-z_j}$$

Then

$$A_{1j} = \frac{1}{(k-j)!}\left\{\frac{d^{k-j}}{dz^{k-j}}(z-z_r)^k \frac{X(z)}{z}\right\}\Bigg|_{z=z_r}, \; j = 1, 2, 3, \dots, k$$

$$A_j = \left\{(z-z_j)\frac{X(z)}{z}\right\}\Bigg|_{z=z_j}, \; j = k+1, k+2, \dots, M$$

Example: Find $x(n)$) for $X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$

Solution: $X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} = \frac{z^3}{(z+1)(z-1)^2}$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_{12}}{(z-1)^2} + \frac{A_{11}}{z-1} + \frac{A_3}{z+1}$$

$$A_{12} = \lim_{Z\to 1}\frac{z^2}{(z+1)} = \frac{1}{2}, \qquad A_{11} = \lim_{Z\to 1}\frac{2z(z+1) - z^2}{(z+1)^2} = \frac{3}{4} \qquad A_3 = \lim_{Z\to -1}\frac{z^2}{(z-1)^2} = \frac{1}{4}$$

$$X(z) = \frac{\frac{1}{2}z}{(z-1)^2} + \frac{\frac{3}{4}z}{z-1} + \frac{\frac{1}{4}z}{z+1} \qquad x(n) = \tfrac{1}{2}nu(n) + \tfrac{3}{4}u(n) + \tfrac{1}{4}(-1)^n u(n)$$

Example: Find the inverse Z-transform by division method for

$$X(z) = \frac{z}{3z^2-4z+1} \quad for\ the\ ROC's\ (a)\ |z| > 1, \quad (b)\ |z| < \frac{1}{3}$$

Solution:

a) For ROC $|z| > 1$, we must divide to obtain negative power of z since $|z| > 1$ indicates a right hand sequence.

$$\begin{array}{r|l} & \frac{1}{3}z^{-1} + \frac{4}{9}z^{-2} \\ \hline 3z^2-4z+1 & \cancel{z} \\ & \mp \cancel{z} \pm \frac{4}{3} \mp \frac{1}{3}z^{-1} \\ \hline & \frac{4}{3} - \frac{1}{3}z^{-1} \\ & \mp \frac{4}{3} \pm \frac{16}{9}z^{-1} \mp \frac{4}{9}z^{-2} \\ \hline & \frac{13}{9}z^{-1} - \frac{4}{9}z^{-2}\ \text{etc} \end{array}$$

$X(z) = \frac{1}{3}z^{-1} + \frac{4}{9}z^{-2} + \ldots$as compare with

$$X(z) = \ldots + x(-1)\,z^1 + x(0)z^0 + x(1)z^{-1} + x(2)\,z^{-2} + \ldots$$

We can recognize that $x(1) = \frac{1}{3}$, $x(2) = \frac{4}{9}$

$$x(n) = \frac{1}{3}\delta(n-1) + \frac{4}{9}\delta(n-2) + \ldots$$

a) For ROC $|z| < \frac{1}{3}$, must divide to get positive power of z since $|z| < \frac{1}{3}$ indicates a left-hand sequence, for negative n.

$$X(z) = z + 4z^2 \Rrightarrow x(-1) = 1\ ,\ x(-2) = 4$$

$$x(n) = \delta(n+1) + 4\delta(n+2) + \ldots$$

$$\begin{array}{r|l} & z + 4z^2 \\ \hline 1 - 4z + 3z^2 & z \\ & \mp z \pm 4z^2 \mp 3z^3 \\ \hline & 4z^2 - 3z^3 \\ & \mp 4z^2 \pm 16z^3 \mp 12z^4 \\ \hline & 13z^3 - 12z^4\ etc \end{array}$$

Notes: At *at ROC* $|Z| > |a|$

- $Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n u(n).$
- $Z^{-1}\left\{\frac{z}{(z-a)^2}\right\} = na^{n-1}u(n)$
- $Z^{-1}\left\{\frac{z}{(z-a)^3}\right\} = \frac{n(n-1)}{2!}a^{n-2}u(n)$
- In general, $Z^{-1}\left\{\frac{z}{(Z-a)^k}\right\} = \frac{n(n-1)(n-2)\ldots(n-k+2)}{(k-1)!}a^{n-k+1}u(n)$

Example : Find the inverse Z-transform by using discrete convolution method for

$$X(z) = \frac{0.632z}{(z-1)(z-0.368)}$$

Solution:

$$X(z) = \frac{0.632z}{(z-1)(z-0.368)} = \frac{0.632z}{(z-1)} \cdot \frac{z^{-1}.z}{(z-0.368)} = X_1(z).X_2(z)$$

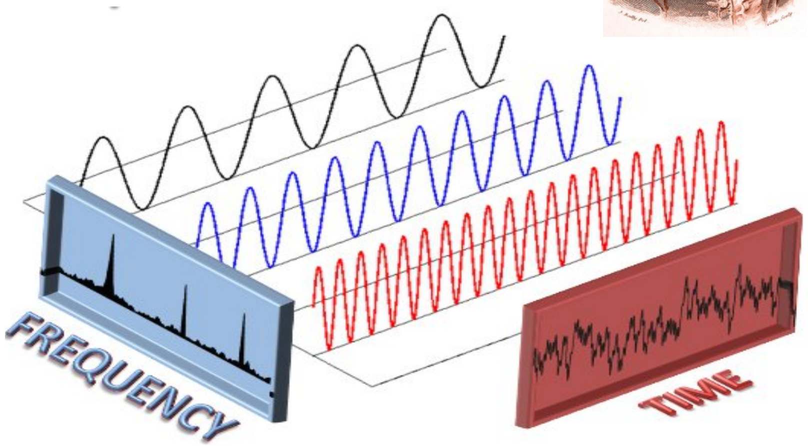
$$x_1(n) = Z^{-1}\{X_1(z)\} = Z^{-1}\left\{\frac{0.632z}{(z-1)}\right\} \Rightarrow x_1(n) = 0.632u(n)$$

$$x_2(n) = Z^{-1}\{X_2(z)\} = Z^{-1}\left\{\frac{z^{-1}.z}{(z-0.368)}\right\} \Rightarrow x_2(n) = (0.632)^{n-1}u(n-1)$$

$$\therefore x(n) = x_1(n) \circledast x_2(n) = 0.632u(n) \circledast (0.632)^{n-1}u(n-1)$$

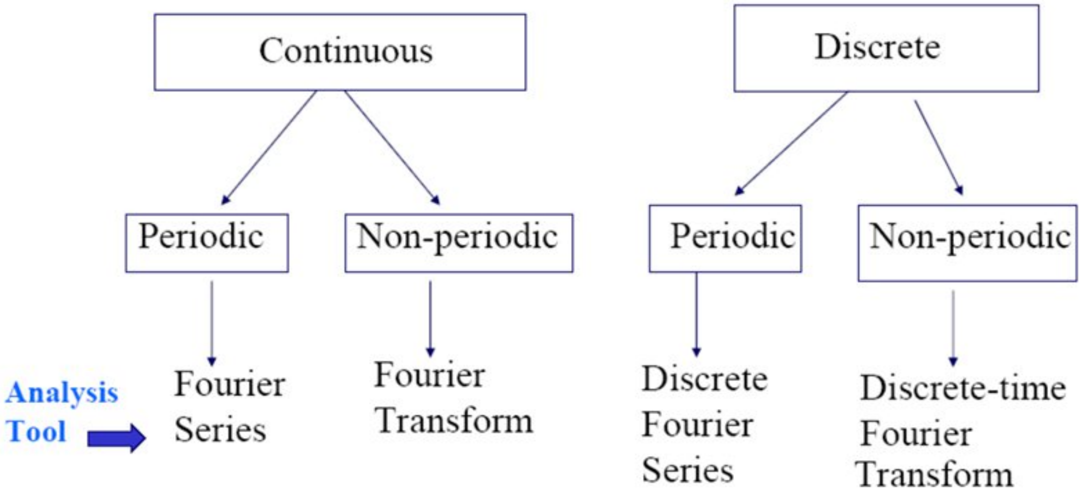
Review of Fourier Analysis

The subject of Fourier analysis is essential for describing certain types of systems and their properties in frequency domain. It is concerned with representing a signal as weighted superposition of complex sinusoids. The complex sinusoids are basic signals that can be used to construct a broad and useful class of signals. There are four distinct Fourier representations:



- ***Continuous-time Fourier series,***
- ***Discrete-time Fourier series,***
- ***Continuous-time Fourier transform,***
- ***Discrete-time Fourier transform.***

Fourier Series (FS) applies to continuous-time periodic signals. Discrete Time Fourier Series (DTFS) applies to discrete-time periodic signals. Nonperiodic signals have Fourier Transform (FT) representation. Discrete Time Fourier Transform (DTFT) applies to a signal that is discrete in time and non-periodic.



Discrete Time Fourier Transform (DTFT) is a frequency domain representation of discrete-time signals that are not necessarily periodic. For discrete-time signals, the **DTFT** of x[n] is given by:-

$$\boxed{X(\Omega) = \mathscr{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\; e^{-j\Omega n}}$$

It is important to note here that ***X(Ω)*** is continuous in frequency and periodic.

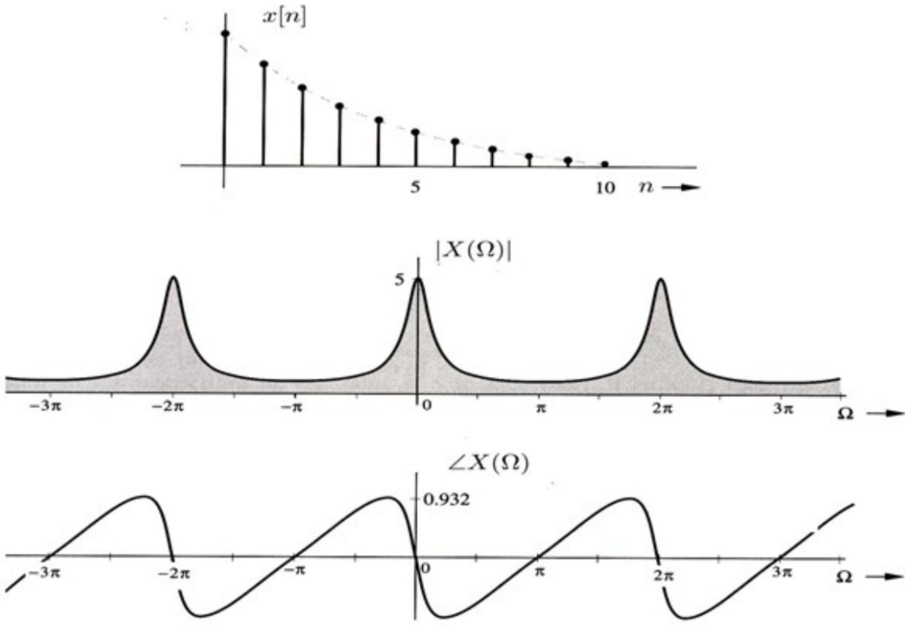
Example: find DTFT for $x(n) = \alpha^n\; u[n] \qquad \alpha < 1$

Solution: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\; e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \alpha^n\; u[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} (\alpha\, e^{-j\Omega})^n = \frac{1}{1-\alpha\, e^{-j\Omega}} = \frac{1}{1-\alpha\cos\Omega + j\alpha\sin\Omega}$

$$|X(\Omega)| = \frac{1}{\sqrt{(1-\alpha\cos\Omega)^2 + (\alpha\sin\Omega)^2}} = \frac{1}{\sqrt{1+\alpha^2-2\alpha\cos\Omega}}$$

The phase is given by

$$arg(X(\Omega)) = -\tan^{-1}\frac{\alpha\sin\Omega}{1-\alpha\cos\Omega}$$



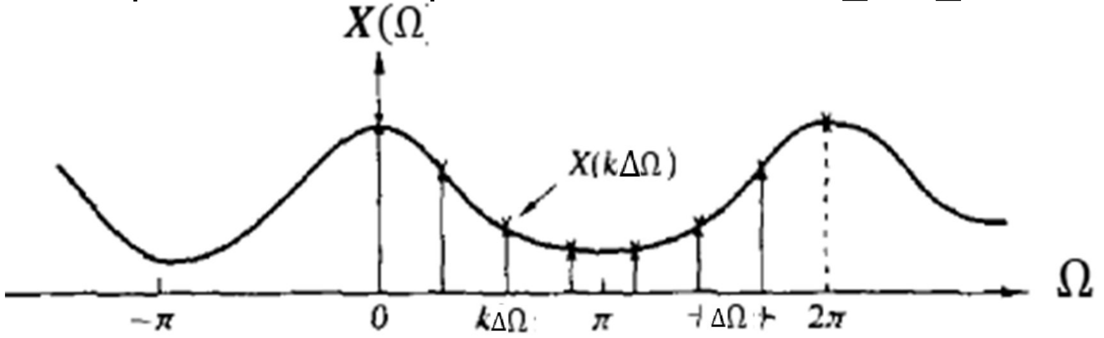
Note that

The spectra are continuous and periodic functions of Ω with the period 2π.

The magnitude spectrum is an even function and the phase spectrum is an odd function of **Ω**

From DTFT to DFT

However, the DTFT is only theoretical analysis, and it is hard to be implemented practically as $X(\Omega)$ is continuous and DTFT is not suitable for DSP applications like digital filtering, image processing becaue DSP processing require a discrete and finit sequences to process. In other words, in DSP, we are able to compute the spectrum only at specific discrete values of Ω. Any signal in any DSP application can be measured only in a finite number of points. As it is known that the spectra $X(\Omega)$ are continuous and periodic functions of Ω with the period 2π. Therefore, it is important to sample the spectrum $X(\Omega)$ in frequency domain so that we obtained what is called **Discrete Fourier Transform (DFT).** Suppose that we sample $X(\Omega)$ periodically in frequency at a spacing of ΔΩ radians between successive samples. Since $X(\Omega)$ is periodic with period 2π, only samples in the fundamental frequency range are necessary. For convenience, we take **N** equidistance samples in the interval $0 \le \Omega \le 2\pi$ with spacing $\Delta\Omega = \frac{2\pi}{N}$, as shown in figure below.

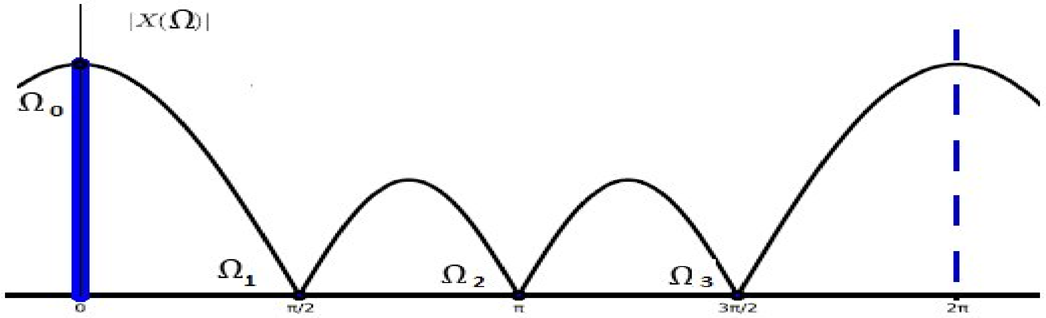


Frequency domain sampling of the Fourier transform

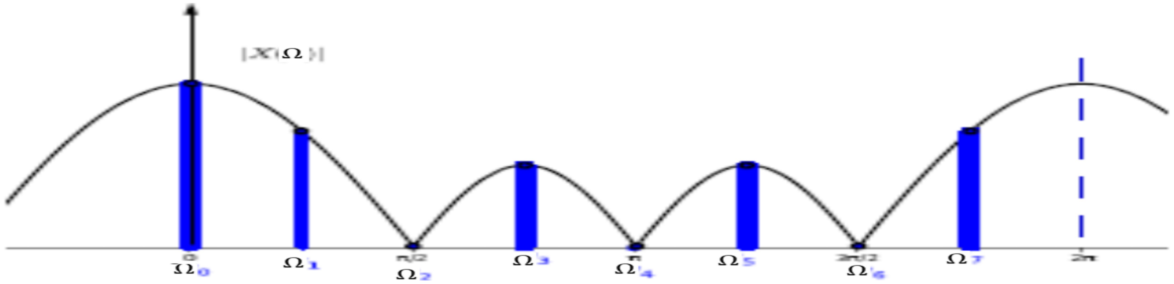
Where $\Delta\Omega = \frac{2\pi}{N}$ is called the ***Spectral reslution***

Example: N = 4 then $\Omega_k = \frac{2\pi k}{4}$, $k \in R_4$,

Spectral Resolution $\Delta\Omega = \frac{\pi}{2}$,



N = 8 then $\Omega_k = \frac{2\pi k}{8}$, $k \in R_8$



The DFT is a formula for transforming a sequence x(n) of length L ≤ N into a sequence of samples X(k) of length N. Let's consider the selection of N, the number of samples in the frequency domain. Then the DFT of x(n) can be given by:

DFT

$$X(k) = X(k\,\Delta\Omega) = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}}$$

k=0, 1, 2, 3...... N-1

This relation is a formula for transforming a sequence x(n) of length L ≤ N into a sequence of frequency samples X(k) of length N .

The inverse discrete Fourier transform (IDFT) is defined as the process of finding the discrete-time sequence x(n) from its frequency response X(k).The inverse DFT is given by:

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi kn}{N}}$$

n=0, 1, 2, 3...... N-1

The formula of DFT and IDFT can be also expressed as

DFT

$$\boxed{X(k) = \sum_{n=0}^{N-1} x[n] \quad W_N^{kn}}$$

k = 0, 1, 2, 3...... N-1

IDFT

$$\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \quad W_N^{-kn}}$$

n=0, 1, 2, 3...... N-1

Where W_N is the twiddle factors which are a set of values that are used to speed up DFT and IDFT calculations. It is given by

$$W_N = e^{-j\frac{2\pi}{N}}$$

This is a matrix of the linear transformation and It is a symmetric matrix.

Note that the computation of each point of the DFT can be accomplished by *N* complex multiplication and *N-1* complex addition. Hence, the **N**-point DFT values can be computed in a total of N^2 complex multiplication and *N(N-1)* complex addition.

It is instructive to view the DFT and IDFT as linear transformation on sequences x(n) and X(k) , respectively. Let us define an N-point vector $\mathbf{x_N}$ of the signal sequence x(n) , n=0, 1, 2,....N-1 and an N-point vector of frequency samples $\mathbf{X_N}$. Also $\mathbf{W_N}$ an be represented as **N×N** matrix and can be written as

$$x_N = \begin{Bmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{Bmatrix}, \qquad X_N = \begin{Bmatrix} X(0) \\ X(1) \\ X(2) \\ . \\ . \\ X(N-1) \end{Bmatrix},$$

$$W_N = \begin{Bmatrix} 1 & 1 & 1 & \ldots\ldots 1 \\ 1 & W_N & W_N^2 \ldots. & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \ldots W_N^{2(N-1)} \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} & \ldots\ldots\ldots \\ & & . & \\ 1 & W_N^{N-1} & \ldots\ldots\ldots & W_N^{(N-1)(N-1)} \end{Bmatrix}$$

With these definitions the N-point DFT may be expressed in matrix form as

$$\boldsymbol{X_N} = \boldsymbol{W_N} \ \boldsymbol{x_N}$$

Where $\boldsymbol{W_N}$ is the matrix of the linear transformation. We observed that $\boldsymbol{W_N}$ is asymmetric matrix. If the inverse of $\boldsymbol{W_N}$ exists, then the above equation can be inverted by multiplying both sided by $\boldsymbol{W_N^{-1}}$. Thus, we obtained

$$\boldsymbol{x_N} = \boldsymbol{W_N^{-1}} \ \boldsymbol{X_N}$$

Example: Calculate the four-point DFT of the aperiodic sequence x[n] ={1, 0, 0, 1}.

Solution:

$$X(k) = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}} \qquad k=0, 1, 2, 3...... N-1$$

For, k=0,

$$X(0) = \sum_{n=0}^{3} x(n) e^{\frac{-j2\pi 0 n}{N}}$$

$$= x(0).e^{\frac{-j2\pi.0.0}{4}} + x(1).e^{\frac{-j2\pi.0.1}{4}} + x(2).e^{\frac{-j2\pi.0.2}{4}} + x(3).e^{\frac{-j2\pi.0.3}{4}} = 2$$

For k=1,

$$X(1) = \sum_{n=0}^{3} x(n) e^{\frac{-j2\pi 1 n}{N}}$$

$$= x(0).e^{\frac{-j2\pi.1.0}{4}} + x(1).e^{\frac{-j2\pi.1.1}{4}} + x(2).e^{\frac{-j2\pi.1.2}{4}} + x(3).e^{\frac{-j2\pi.1.3}{4}}$$

$$= 1.e^{\frac{-j2\pi.1.0}{4}} + 0 + 0 + 1.e^{\frac{-j2\pi.1.3}{4}}$$

$$= 1 + 1.e^{\frac{-j2\pi.1.3}{4}} = 1 + [e^{(\frac{-j3\pi}{2})}] = 1 + [e^{-j270}]$$

$$[\because \pi\ rad = 180°, \therefore \frac{3\pi}{2} rad = \frac{3x180}{2} = 270°$$

$$= 1 + [\cos(270°) - j\sin(270°)]$$

$$[\therefore e^{-j\theta} = \cos\theta - j\sin\theta\ (Euler's Formula)$$

$$= 1 + [0 - (-j1)]$$

$$= 1 + j$$

Complete for k=2 and 3 to find

$$X(k) = [2, 1+j, 0, 1-j]$$

Example: Compute the 4-point DFT of the aperiodic sequence x[n]={0, 1, 2, 3} of length N = 4?

Solution: The first step is to determine the matrix $\mathbf{W_4}$ by exploiting the periodicity property of and the symmetry property

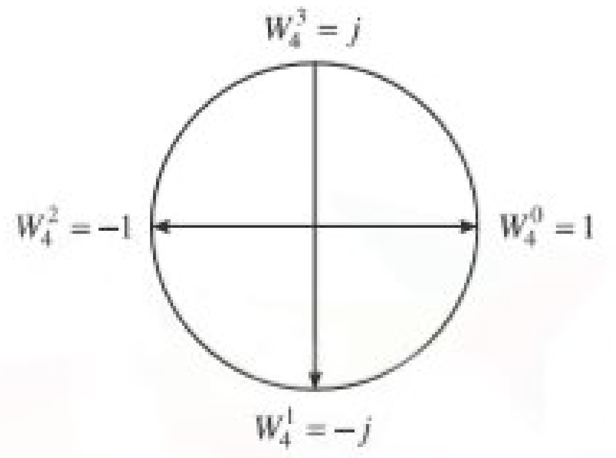
$$\mathbf{W}_N^{k+\frac{N}{2}} = -\mathbf{W}_N^k$$

$$W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$$

$$W_4^0 = e^{-j\frac{2\pi}{4}\times 0} = 1,$$

$$W_4^1 = e^{-j\frac{\pi}{2}\times 1} = cos\left(-\frac{\pi}{2}\right) + jsin\left(-\frac{\pi}{2}\right) = -j,$$

$$W_4^2 = -1, \qquad W_4^3 = j$$



(a) 4-point DFT

$$\mathbf{W}_4 = \begin{bmatrix} \mathbf{W}_4^0 & \mathbf{W}_4^0 & \mathbf{W}_4^0 & \mathbf{W}_4^0 \\ \mathbf{W}_4^0 & \mathbf{W}_4^1 & \mathbf{W}_4^2 & \mathbf{W}_4^3 \\ \mathbf{W}_4^0 & \mathbf{W}_4^2 & \mathbf{W}_4^4 & \mathbf{W}_4^6 \\ \mathbf{W}_4^0 & \mathbf{W}_4^3 & \mathbf{W}_4^6 & \mathbf{W}_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathbf{W}_4^1 & \mathbf{W}_4^2 & \mathbf{W}_4^3 \\ 1 & \mathbf{W}_4^2 & \mathbf{W}_4^0 & \mathbf{W}_4^2 \\ 1 & \mathbf{W}_4^3 & \mathbf{W}_4^2 & \mathbf{W}_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix}$$

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} [0\ 1\ 2\ 3] = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

The DFT X(k) of the sequence $x(n)$ is given by

DFT

$$X(k) = \sum_{n=0}^{N-1} x[n] \quad W_N^{kn} \qquad k = 0, 1, 2, 3...... N-1$$

and the inverse DFT is

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \quad W_N^{-kn} \qquad n=0, 1, 2, 3...... N-1$$

Where the twiddle factor is $W_N = e^{-j\frac{2\pi}{N}}$

Since the DFT and IDFT involve basically the same type of computations. It can be observed from these equations that the determination of each $X(k)$ requires N complex multiplications and $N-1$ complex additions. Since we have to calculate $X(k)$ for k=0,1,…..,$N-1$. It follows that the direct computation of the DFT requires N^2 complex multiplication and $N(N-1)$ complex additions. Thus, procedures that reduce the calculation burden are of considerable interest. These procedures are known **as fast Fourier transform (FFT) algorithms.**

The basic idea of the FFT is to divide the sequence into sub-sequences of smaller length and then combine these smaller DFTs suitably to obtain the DFT of the original sequence. Assuming that data length N is a power of 2, so that N is of the form $N=2^m$, where m is positive integer. (i.e. N=2,4,8,……etc). Accordingly the algorithm is referred to as **radix-2 algorithm**. The FFT algorithm exploits the symmetry and periodicity properties of the phase factor W_N. In particular, these three properties are:

Periodicity property: $W_N^{k+N} = W_N^k$ $\qquad$ ***Example*** : $N = 8$, *then* $W_8^0 = W_8^8 = W_8^{16} = ...$

$W_N^{k+\frac{N}{2}} = -W_N^k$ $\qquad$ ***Example***: $N = 8$, *then* $W_8^0 = 1$ *then* $W_8^4 = -1$ and $W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$ *then* $W_8^5 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

Symmetry property:

$W_N^m = W_{\frac{N}{m}}$ $\qquad$ ***Example*** : $N = 4$, $m = 2$, *then* $W_4^2 = W_2$