Example 4: Find the Z-transform including the region of convergence of

1) $x(n) = \delta(n)$ 2) $x(n) = \delta(n-k)\ \ k > 0$ 3) $x(n) = u(n)$

Solution:

1) $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \{\delta(n)\} z^{-n} = z^0 = 1,$ $ROC\ is\ ALL\ values\ of\ z$

2) $X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \{\delta(n-k)\} z^{-n} = \sum_{m=-\infty}^{\infty} \{\delta(m)\} z^{-m-k}$

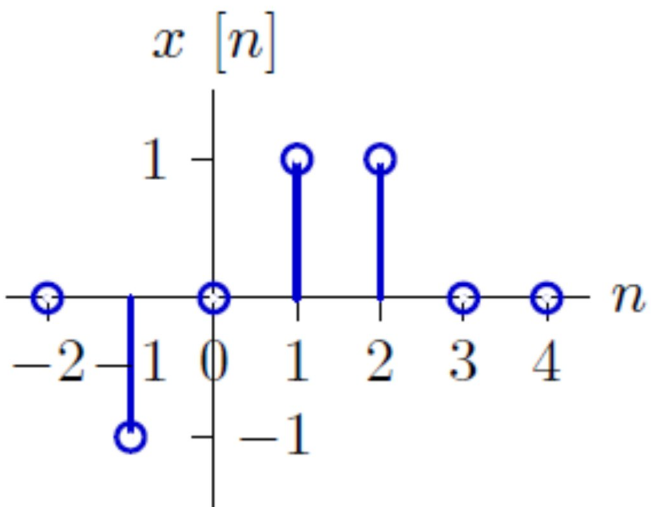
$= z^{-k} \sum_{m=-\infty}^{\infty} \{\delta(m)\} z^{-m} = z^{-k},$ $ROC\ \ |z| > 0$

3)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \{u(n)\} z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$$

$$Z\{u(n)\} = \frac{z}{z-1}\ \ , ROC\ \ |z| > 1$$

Example 5: Find the Z-transform including the region of convergence of



Solution: x(n)=[-1, 0, 1, 1]
↑ (under 0)

$$x(n) = -\delta(n+1) + \delta(n-1) + \delta(n-2)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = x[-1]\ z + x[1] z^{-1} + x[2] z^{-2}$$

$$X(z) = -z + z^{-1} + z^{-2}$$

$ROC\ 0 < |z| < \infty$

Z- Transform (ZT) Table

Line No.	$x(n)$, $n \geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$	
2	$\delta(n)$	1	$\lvert z\rvert > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\lvert z\rvert > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
5	$n^2u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
7	$e^{-na}u(n)$	$\frac{z}{(z-e^{-a})}$	$\lvert z\rvert > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
9	$\sin(an)u(n)$	$\frac{z\sin(a)}{z^2 - 2z\cos(a) + 1}$	$\lvert z\rvert > 1$
10	$\cos(an)u(n)$	$\frac{z[z-\cos(a)]}{z^2 - 2z\cos(a) + 1}$	$\lvert z\rvert > 1$
11	$a^n \sin(bn)u(n)$	$\frac{[a\sin(b)]z}{z^2 - [2a\cos(b)]z + a^2}$	$\lvert z\rvert > \lvert a\rvert$
12	$a^n \cos(bn)u(n)$	$\frac{z[z - a\cos(b)]}{z^2 - [2a\cos(b)]z + a^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
13	$e^{-an}\sin(bn)u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\lvert z\rvert > e^{-a}$
14	$e^{-an}\cos(bn)u(n)$	$\frac{z[z - e^{-a}\cos(b)]}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\lvert z\rvert > e^{-a}$

1. Linearity

If $X_1(z) = Z\{x_1(n)\}$ *with ROC* $R_{1-} < |z| < R_{1+}$

$X_2(z) = Z\{x_2(n)\}$ *with ROC* $R_{2-} < |z| < R_{2+}$

$$\boldsymbol{Z\{a_1x_1(n) + a_2x_2(n)\} = a_1X_1(z) + a_2X_2(z)}$$

$$\boldsymbol{ROC = ROC_1 \cap ROC_2 \{Intersection\}}$$

2. Shifting

$$\boldsymbol{Z\{x(n-n_o)\} = z^{-n_o}X(z)}$$

ROC no change

Example: Find the Z-transform including the region of convergence of

$$\boldsymbol{x(n) = u(n-4)}$$

Solution:

$$X(z) = Z\{u(n-4)\} = z^{-4}Z\{u(n)\} = z^{-4}\frac{z}{z-1} = \frac{z^{-3}}{z-1}$$

, $ROC \quad |z| > 1$

3. Multiplication by an exponential

$$Z\{a^n x(n)\} = X(z)|_{Z \to \frac{Z}{a}} = X\left(\frac{z}{a}\right) \qquad \text{with ROC } |a|\, R_{x-} < |z| < |a|\, R_{x+}$$

$$Z\{e^{\pm an} x(n)\} = X(z)|_{z \to ze^{\mp a}} = X(ze^{\mp a}) \qquad \text{with ROC } |e^{\pm a}|\, R_{x-} < |z| < |e^{\pm a}|\, R_{x+}$$

Example: Find the Z-transform including the region of convergence of $x(n) = ne^{an}u(n)$

Solution:

$$Z\{nu(n)\} = \sum_{n=-\infty}^{\infty} nu(n)\, z^{-n} = \sum_{n=0}^{\infty} nz^{-n} = \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$$

$$ROC \ |Z^{-1}| < 1 \to |Z| > 1$$

$$\because Z\{e^{an}x(n)\} = X\left(\frac{z}{e^a}\right) \qquad \therefore Z\{ne^{an}u(n)\} = \frac{ze^{-a}}{(ze^{-a}-1)^2} \quad , ROC \ |z| > e^a$$

4. Multiplication by n (Ramp)

$$Z\{nx(n)\} = -z\frac{dX(z)}{dz} \quad , ROC\ no\ change$$

Example : Find the Z-transform including the region of convergence of $Z\{ne^{an}u(n)\}$

Solution: $\because Z\{e^{an}u(n)\} = \frac{z}{z-e^{a}}$ $\quad and \quad$ $Z\{nx(n)\} = -z\frac{dX(z)}{dz}$

$$Z\{ne^{an}u(n)\} = -z\frac{d\left(\frac{z}{z-e^{a}}\right)}{dz} = -z\frac{(z-e^{a}).1-z.1}{(z-e^{a})^2} = \frac{ze^{a}}{(z-e^{a})^2}$$

5. Discrete convolution

$$x_1(n) \circledast x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$Z\{x_1(n) \circledast x_2(n)\} = X_1(z)\ X_2(z) \quad ,with\ ROC,\ Z \in R_1 \cap R_2$$

Proof.

$$Z\{x_1(n) \circledast x_2(n)\} = \sum_{n=-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right\} Z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\left\{\sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-n}\right\} = X_2(z)\sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} = X_1(z)\quad X_2(z)$$

6. Initial value theorem

$$x(0) = \lim_{n\to 0} x(n) = \lim_{z\to\infty} X(z)$$

7. Final value theorem

$$x(\infty) = \lim_{n\to\infty} x(n) = \lim_{z\to 1}(1-z^{-1})X(z)$$

8. Time reversal

$$x(n) \leftrightarrow X(z) \quad , ROC \quad r_1 < |z| < r_2$$

$$x(-n) \leftrightarrow X\left(z^{-1}\right) \quad ,ROC \quad \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Example: : Find Z{-n u(-n)}

Solution: $Z\{n\,u(n)\} = \frac{z}{(z-1)^2}$ $\quad ROC, Z\in |Z| > 1$

$\rightarrow Z\{-n\,u(-n)\} = \frac{1/z}{(1/z-1)^2}$ $\quad ROC, Z\in |Z| > 1$

Example: Find the Z-transform including the region of convergence of $x(n) = cos(\omega_o n)u(n)$

Solution $Z\left\{\frac{e^{j\omega_o n} + e^{-j\omega_o n}}{2} u(n)\right\} = \frac{1}{2} Z\left\{e^{j\omega_o n} u(n) + e^{-j\omega_o n} u(n)\right\} = \frac{1}{2}\left[\frac{z}{z - e^{j\omega_o}} + \frac{z}{z - e^{-j\omega_o}}\right]$

$$= \frac{1}{2}\left[\frac{z^2 - ze^{-j\omega_o} + z^2 - ze^{j\omega_o}}{(z - e^{j\omega_o})(z - e^{-j\omega_o})}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{j\omega_o} + e^{-j\omega_o})}{z^2 - ze^{-j\omega_o} - ze^{-j\omega_o} + 1}\right] = \frac{1}{2}\left[\frac{2z^2 - 2z\cos(\omega_o)}{z^2 - z(e^{-j\omega_o} + e^{-j\omega_o}) + 1}\right]$$

$$= \left[\frac{z^2 - z\cos\omega_o}{z^2 - 2z\cos\omega_o + 1}\right], \quad ROC\ |z| > \left|e^{\pm j\omega_o}\right| \rightarrow |z| > 1$$

Example: Find the z-transform and its ROC of the following sequence: x[n]={4, 2, -1, 0, 3, -4} (arrow under 4)

Solution: The **z**-transform *X*(*z*) of *x*(*n*) is given by

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-2}^{3} x(n)\, z^{-n}$$

$$= x(-2)\ z^2 + x(-1)\ z^1 + x(0) + x(2)\ z^{-2} + x(3)\ z^{-3}$$

$$= 4\ z^2 + 2\ z\ - 1 + 3\ z^{-2} - 4\ z^{-3}$$

Therefore, *X*(*z*) will be finite if and only if z is not equal to 0 or ∞. Its ROC is given by 0 < |z| < ∞.

Example: Find the Z-transform including the region of convergence of

$$x(n) = [\,3(2^n) - 4\,(3^n)]\,u(n)$$

Solution.

Assume that $x_1(n) = (2^n)\,u(n)$ and $x_2(n) = (3^n)\,u(n)$ then the signal x(n) can be written as

$$x(n) = 3x_1(n) - 4\,x_2(n) \ldots.$$

According to linear property the ZT will be

$$X(z) = 3X_1(z) - 4\,X_2(z) \ldots.$$

We know that [$(a^n)\,u(n) \xrightarrow{\quad Z \quad} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$ $\quad$ *with ROC* $|z| > |a|$

$$X_1(z) = \frac{z}{z-2} \qquad \text{with ROC } |z| > 2$$

$$X_2(z) = \frac{z}{z-3} \qquad \text{with ROC } |z| > 3$$

Therefore, the overall z-transform will be

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}} = \frac{3z}{z-2} - \frac{4z}{z-3} \qquad \text{with ROC } |z| > 3$$

Example: Find the Z-transform including the region of convergence of

$$x(n) = (n-2)a^{(n-2)}\cos(\omega_o(n-2))u(n-2)$$

Use $Z\{\cos(\omega_o n)u(n)\} = \frac{z^2 - z\cos\omega_o}{z^2 - 2z\cos\omega_o + 1}$ and any necessary properties.

Solution:

$$Z\{x(n)\} = z^{-2}\, Z\{n\, a^{(n)}\cos(\omega_o n)u(n)\} \quad \textit{by using shift property}$$

$$= z^{-2}\left\{-z\frac{d}{dz}Z\{a^{(n)}\cos(\omega_o n)u(n)\}\right\} \quad \textit{by using multiplication by n property}$$

$$= -z^{-1}\left\{\frac{d}{dz}Z\{\cos(\omega_o n)u(n)\}\Big|_{z\to\frac{z}{a}}\right\} \quad \textit{by using multiplication by } a^n \textit{ property}$$

$$ROC\ \ |Z| > |a|,$$

$$Z\{\cos(\omega_o n)u(n)\} = \frac{z^2 - z\cos\omega_o}{z^2 - 2z\cos\omega_o + 1}$$

$$Z\{\cos(\omega_o n)u(n)\}\Big|_{z\to\frac{z}{a}} = \frac{a^{-2}z^2 - a^{-1}z\cos\omega_o}{a^{-2}z^2 - 2a^{-1}z\cos\omega_o + 1} = \frac{z^2 - az\cos\omega_o}{z^2 - 2az\cos\omega_o + a^2}$$

$$Z\{x(n)\} = -z^{-1}\frac{d}{dz}\left(\frac{z^2 - az\cos\omega_o}{z^2 - 2az\cos\omega_o + a^2}\right) = \ldots$$

Example : Determine the z-transform of the signal

$$x[n] = \begin{cases} 1, & n = -1 \\ 2, & n = 0 \\ -1, & n = 1 \\ 1, & n = 2 \\ 0, & otherwise \end{cases}$$

Use the z-transform to determine the DTFT of *x* [redacted]*n*[redacted] .

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-1}^{2} x[n]z^{-n} = x[-1]z^{1} + x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2}$$

$$= (1)z^{1} + (2)z^{0} + (-1)z^{-1} + (1)z^{-2}$$

$$= z + 2 - z^{-1} + z^{-2}$$

We obtain the DTFT from $X(z)$ by substituting $z = e^{j\Omega}$:

$$X(e^{j\Omega}) = e^{j\Omega} + 2 - e^{-j\Omega} + e^{-j2\Omega}$$

The inverse Z-transformation is used to convert signals/systems from frequency domain (Z-domain) to the discrete time domain x(n). When using the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

It is often useful to be able to find x[n] given X(z) . This can be done by taking the inverse of Z-transform

$$x(n) = Z^{-1}\{X(z)\} = \frac{1}{2\pi}\int X(z)\, z^{n-1}\, dz$$

we will not evaluate the complex contour integral for the inverse z-transform directly. Instead we will use one of the following techniques to find the inverse of Z-transform

1. Inspection
2. Partial-Fraction Expansion
3. Power Series Expansion
4. Discrete convolution

This "method" is to basically become familiar with thez-transform pair tablesand then "reverse engineer". In order to find the inv z-transform, we compare X(z) to one of the standards transform pairs listed in z-transform pair tables and and of the properties of the Z-Transform

Example : When given $X(z) = \frac{z}{z-\alpha}$ with an ROC of $|z| > \alpha$

Solution: we could determine "by inspection" that $x[n]=\alpha^n u[n]$

Example find the inverse z-transform of $Y(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3}$

Solution : since $Az^{-m} = A\,\delta(n-m)$

$$y(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 2\delta(n-3) = [1, 2, 3\ 2]$$

$$\textit{Example}\ \ Z\{-b^n u(-n-1)\} = \frac{z}{z-b}\ \ ,ROC\ \ |z| < b \quad \rightarrow \quad Z^{-1}\left\{\frac{z}{z-b}\right\} = -b^n u(-n-1)$$

$$\textit{Example}\ \ Z\{u(n)\} = \frac{z}{z-1}\ \ ,ROC\ \ |z| > 1 \quad \rightarrow \quad Z^{-1}\left\{\frac{z}{z-1}\right\} = u(n)$$

$$\textit{Example}\ \ Z\{-u(-n)\} = \frac{z}{z-1}\ \ ,ROC\ \ |z| < 1 \quad \rightarrow \quad Z^{-1}\left\{\frac{z}{z-1}\right\} = -u(-n-1)$$

Example $Z\{\delta(n)\} = 1$,*ROC all z* $\rightarrow$ $Z^{-1}\{1\} = \delta(n)$

Example $Z\{nu(n)\} = \frac{z}{(z-1)^2}$ ROC: $|Z| > 1$ $\rightarrow$ $Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = n\,u(n)$

Example: Determine the signal $x(n)$ for $X(z) = \ln(1 + az^{-1}),\ |z| > |a|$

Solution:

using $Z\{nx(n)\} = -z\frac{d}{dz}X(z)$, $\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}$

$$\frac{d}{dz}X(z) = \frac{1}{1 + az^{-1}}\left(-az^{-2}\right) \rightarrow -z\frac{d}{dz}X(z) = \frac{\left(az^{-1}\right)}{1 + az^{-1}}$$

$$ROC\ \left|-az^{-1}\right| < 1 \rightarrow |z| > |a|$$

Take the inverse of Z-transform for both sides to get:

$$nx(n) = Z^{-1}\left\{\frac{\left(az^{-1}\right)}{1 + az^{-1}}\right\} \rightarrow nx(n) = a(-a)^{n-1}u(n-1)$$

$$\therefore x(n) = \frac{1}{n}(-1)^{n-1}a^n u(n-1)$$