Form of Particular Solutions Corresponding to Commonly Used Inputs.

Continuous Time		Discrete Time	
Input	Particular Solution	Input	Particular Solution
1	c	1	c
t	$c_1 t + c_2$	n	$c_1 n + c_2$
e^{-at}	ce^{-at}	α^n	$c\alpha^n$
$\cos(\omega t + \phi)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$	$\cos(\Omega n + \phi)$	$c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$

Notes:

1- Forced response of the system = particular solution (usually has the form of the input signal).

2- Natural response of the system = homogeneous solution in the form $\mathbf{A}e^{st}$ (depends on initial conditions & forced response).

3- For causal LTI systems defined by linear constant coefficient differential equations, the initial conditions are always

$$y(0) = \frac{dy(0)}{dt} = \dots = \frac{dy^{N-1}(0)}{dt^{N-1}} = 0$$

which is called **initial rest**.

Example: Solve the LTI system described by $\frac{dy(t)}{dt} + 2y(t) = x(t)$ where $x(t) = Ke^{3t}\,u(t)$ where K is a real number.

Solution $y(t) = y_p(t) + y_h(t)$

$y_p(t)$ has the same form as $x(t)$, that is $y_p(t) = Ce^{3t}$

Substitute $x(t)$ and $y_p(t) = C\,e^{3t}$ in $\frac{dy(t)}{dt} + 2y(t) = x(t)$, we get

$$3Ce^{3t} + 2C\,e^{3t} = Ke^{3t}$$

Cancelling the factor e^{3t} from both sides we get $C = K/5$, so that => $y_p(t) = \frac{K}{5}e^{3t}$

Now substituting $y_h(t) = Ae^{st}$ into $\frac{dy(t)}{dt} + 2y(t) = 0$, to get

$A\,s\,e^{st} + 2Ae^{st} = 0$, which holds for **s** = -2, to obtain => $y_h(t) = Ae^{-2t}$

Combining the natural response and the forced response to get the general solution

$$y(t) = y_p(t) + y_h(t) = \frac{K}{5}e^{3t} + Ae^{-2t}$$

For LTI system, the initial rest implies that $y(0) = 0 = \frac{K}{5} + A$ => $A = -\frac{K}{5}$

Then the final solution will be

$$y(t) = \frac{K}{5}(e^{3t} - e^{-2t}) \qquad \text{for } t > 0$$

A difference equation is the discrete-time analogue of a differential equation. We simply use differences *($x[n]-x[n-1]$)* rather than derivatives *($dx(t)/dt$)*. The discrete-time counterpart of the general differential equation is the N^{th} order linear constant-coefficient difference equation given by

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

where coefficients a_k, and b_k, are real constants. The order N refers to the largest delay of $y[n]$.

Linear Constant-Coefficient Difference Equations

- For a general DT LTI system, with N-th order,

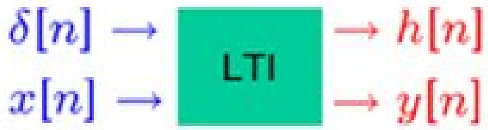
$$x[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow y[n]$$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\Rightarrow \sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

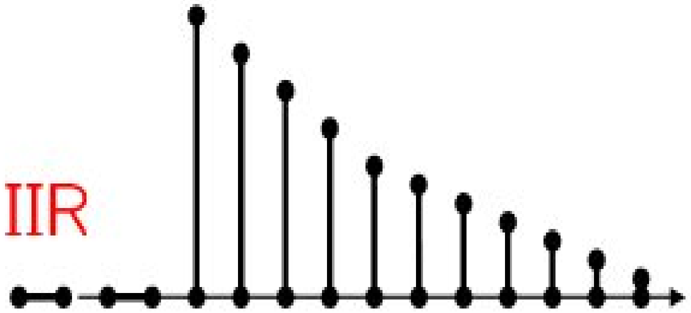
Example Solve the following difference equation $y[n]-\frac{1}{2}y[n-1]=x[n]$ where $x[n]=K\delta[n]$ and assume initial rest.



- **Recursive Equation:**
  - For example,

$$y[n]-\frac{1}{2}y[n-1]=x[n] \qquad \begin{aligned} y[n] &= 0, \quad \text{for } n \le -1 \\ x[n] &= K\,\delta[n] \end{aligned}$$

$$\Rightarrow \begin{cases} y[0]=x[0]+\frac{1}{2}y[-1] & = K \\ y[1]=x[1]+\frac{1}{2}y[0] & = K\ \frac{1}{2} \\ y[2]=x[2]+\frac{1}{2}y[1] & = K\ \left(\frac{1}{2}\right)^2 \\ \vdots & \\ y[n]=x[n]+\frac{1}{2}y[n-1] & = K\ \left(\frac{1}{2}\right)^n \end{cases}$$



$$\Rightarrow h[n]=\left(\frac{1}{2}\right)^n u[n]$$

$\Rightarrow$ an Infinite Impulse Response (IIR) system

It should also be noted that, in example above, the impulse response $h(n)$ contains an infinite number of terms in its duration due to the past output term $y(n-1)$. Such a system is called an ***infinite impulse response (IIR) system***.

Example: Given the linear time-invariant system $y(n) = 0.5x(n) + 0.25\,x(n-1)$

Determine the impulse response $h(n)$. Write the output using the obtained impulse response.

Solution: let $x(n) = \delta(n)$, then $h(n) = 0.5\,\delta(n) + 0.25\,\delta(n-1)$

Thus, for this particular linear system, we have $h(n) = \begin{cases} 0.5 & n = 0 \\ 0.25 & n = 1 \\ 0.0 & \text{otherwise} \end{cases}$

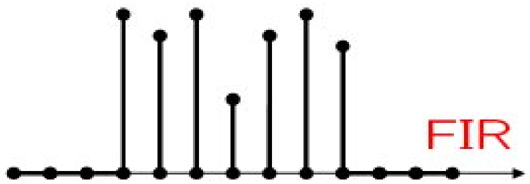
The output response can be written as $y(n) = h(0)x(n) + h(1)\,x(n-1)$

- Nonrecursive Equation: $\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$
  - When N = 0,

$$\Rightarrow y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$\Rightarrow h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$



$\Rightarrow$ a Finite Impulse Response (FIR) system

From this result, it could be noted that if the difference equation without the past output terms, $y(n-1), y(n-2), \ldots, y(N)$, that is the corresponding coefficients $a_1, a_2, \ldots, a_N$, are zeros, the impulse response $h(n)$ has a finite number of terms. We call this a ***finite impulse response (FIR) system***.

Example (Second-order system) : Find the first three terms in output response for system described by

$$y[n] - 1.5y[n-1] + y[n-2] = 2u[n-2]$$

Initial conditions $y[-2] = 2, \quad y[-1] = 1$ input $u[n]$

Solution

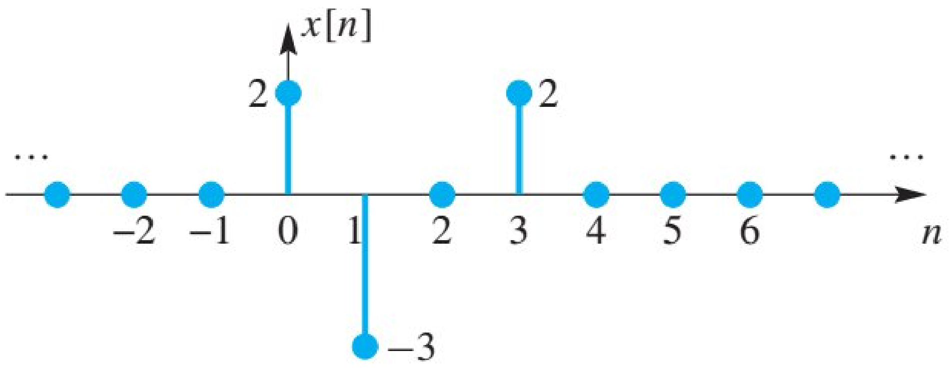
$$y[n] = 1.5y[n-1] - y[n-2] + 2u[n-2]$$

$$\begin{aligned}
y[0] &= 1.5y[-1] - y[-2] + 2u[-2] \\
&= 1.5 \times 1 - 1 \times 2 + 2 \times 0 = -0.5 \\
y[1] &= 1.5y[0] - y[-1] + 2u[-1] \\
&= 1.5 \times (-0.5) - 1 \times 1 + 2 \times 0 = -1.75 \\
y[2] &= 1.5y[1] - y[0] + 2u[0] \\
&= 1.5 \times (-1.75) - 1 \times (-0.5) + 2 \times 1 = -.125 \\
y[3] &= 1.5y[2] - y[1] + 2u[1] \\
&= 1.5 \times (-0.125) - 1 \times (-1.75) + 2 \times 1 = 3.5625
\end{aligned}$$

Example: Find and plot the output response for the system described by the following DE

$$\boxed{y[n] - 0.8y[n-1] - 5\,x[n]=0}$$

Assume that the input signal is $\boxed{x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]}$ with rest conditions.



Solution: Rest conditions: $y[n] = 0$ for $n < 0$.

$$\boxed{y[n] = 0.8y[n-1] + 5\,x[n]}$$

$y[0] = 0.8\,y[-1] + 5\,x(0) = 0.8(0) + 5(2) = 10$

$y[1] = 0.8\,y[0] + 5\,x[1] = 0.8(10) + 5(-3) = -7$

$y[2] = 0.8\,y[1] + 5\,x[2] = 0.8(-7) + 5(0) = -5.6$

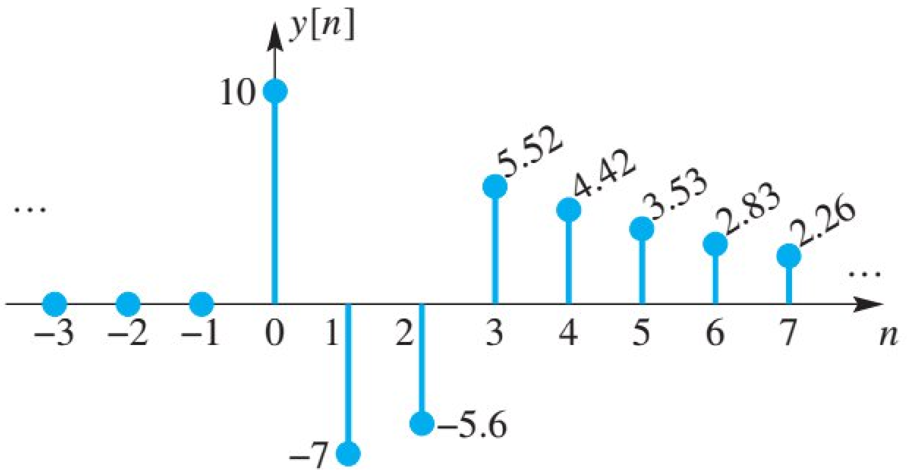
$y[3] = 0.8\,y[2] + 5\,x[3] = 0.8(-5.6) + 5(2) = 5.52$

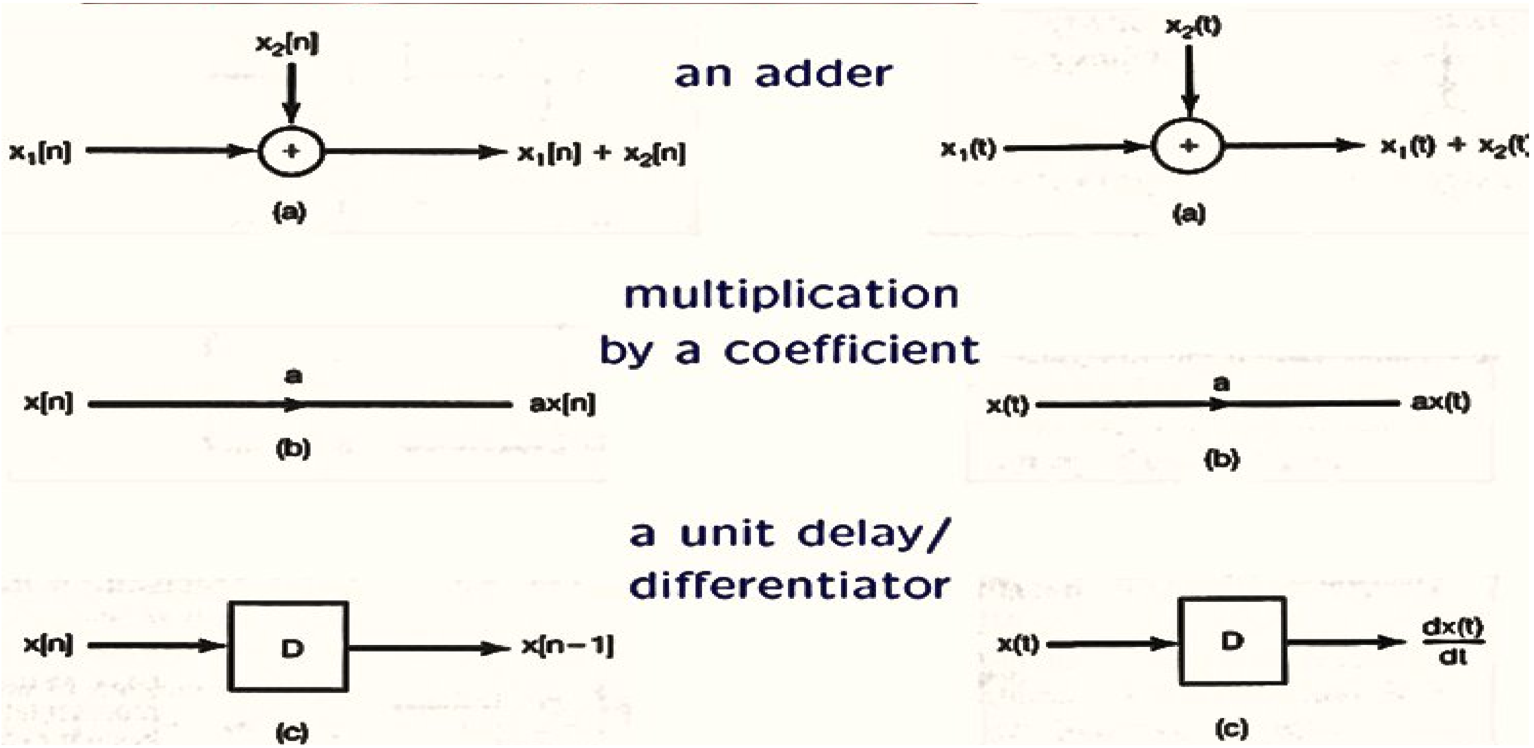
$y[4] = 0.8\,y[3] + 5\,x[4] = 0.8(5.52) + 5(0) = 4.42$

$y[5] = 0.8\,y[4] + 5x[5] = 0.8(4.416) + 0 = 3.53$

$y[6] = 2.83$

$y[7] = 2.26$





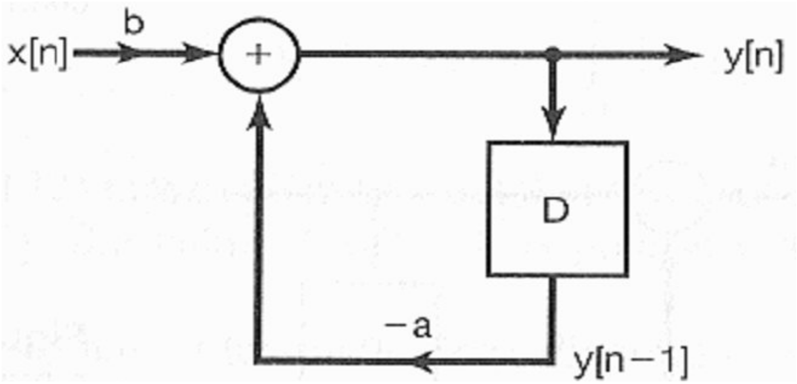
Example: Represent the following Equations by block diagram

$$y[n] + ay[n-1] = bx[n]$$

$$\frac{d}{dt}y(t) + ay(t) = bx(t)$$

Solution:

$$y[n] = -ay[n-1] + bx[n]$$



Block diagram for *y(n)+a y(n-1)=b x(n)*

Check yourself for $y(t) = -\frac{1}{a}\frac{d}{dt}y(t) + \frac{b}{a}x(t)$

The z-transform is a very important tool in describing and analyzing systems. It also offers the techniques for digital filter design and frequency analysis of DT signals. It is an Extension of DTFT but can be applied to a broader class of signals than DTFT. It used to evaluate the stability and causality of LTI system (this can be done by analysing the roots of the polynomial). ZT transforms a discrete-time signal (in time domain) to frequency z-plane (polynomial).
The Z-transform of the discrete signal x(n) is expressed as:

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The **z** in the above definition is a complex variable $z = e^{j\Omega}$. Recall the definition of DTFT.

$$X(\Omega) = \mathscr{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\; e^{-j\Omega\, n}$$

$$\therefore\ X(z)\big|_{z=e^{j\Omega}} = DTFT\ of\ x[n]$$

Note that for **causal sequence**, the summation taken from $n = 0$ to $n = \infty$

$$X(z) = \sum_{n=0}^{\infty} x(n)\; z^{-n} = x(0)\; z^{0} + x(1)\; z^{-1} + x(2)\; z^{-2} + \;.....$$

Region of Convergence (ROC)

A region on the z-plane where the magnitude of $X(z)$ is not ∞ where

$$|X(z)| \leq R \neq \infty$$

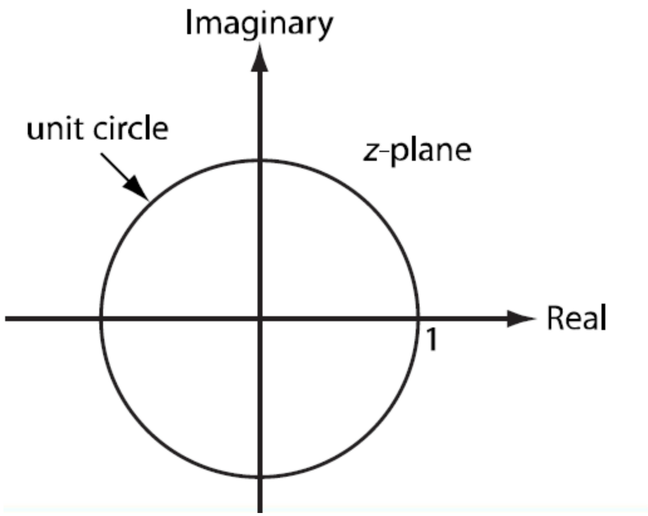
Converge means $X(z)$ has a true value which is not infinity. In other words, $X(z)$ exists.

- Boundary of ROC is a circle centered at $z=0$
- Only poles will determine the ROC where zeros gives no effect to the ROC
- ROC is a region between poles. Thus, there must be no poles inside ROC
- ROC is a connected region. Thus, only three shapes of ROC are valid, or else there will be no ROC for the system.

- In general, this region is bounded by

$$R_{x-} < |z| < R_{x+}$$

R_{x-} is lower (minimum) limit of this region may be zero,
R_{x+} is upper (maximum) limit of it may be infinity.



Example 1: Find the Z-transform including the region of convergence of

$$x(n) = a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Positive region

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \{a^n u(n)\} z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

By using the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x}, \quad |x| < 1$$

$$\therefore X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

The above result converges if $\left|az^{-1}\right| < 1 \; or \; |z| > |a|$ **ROC: $|z| > |a|$**

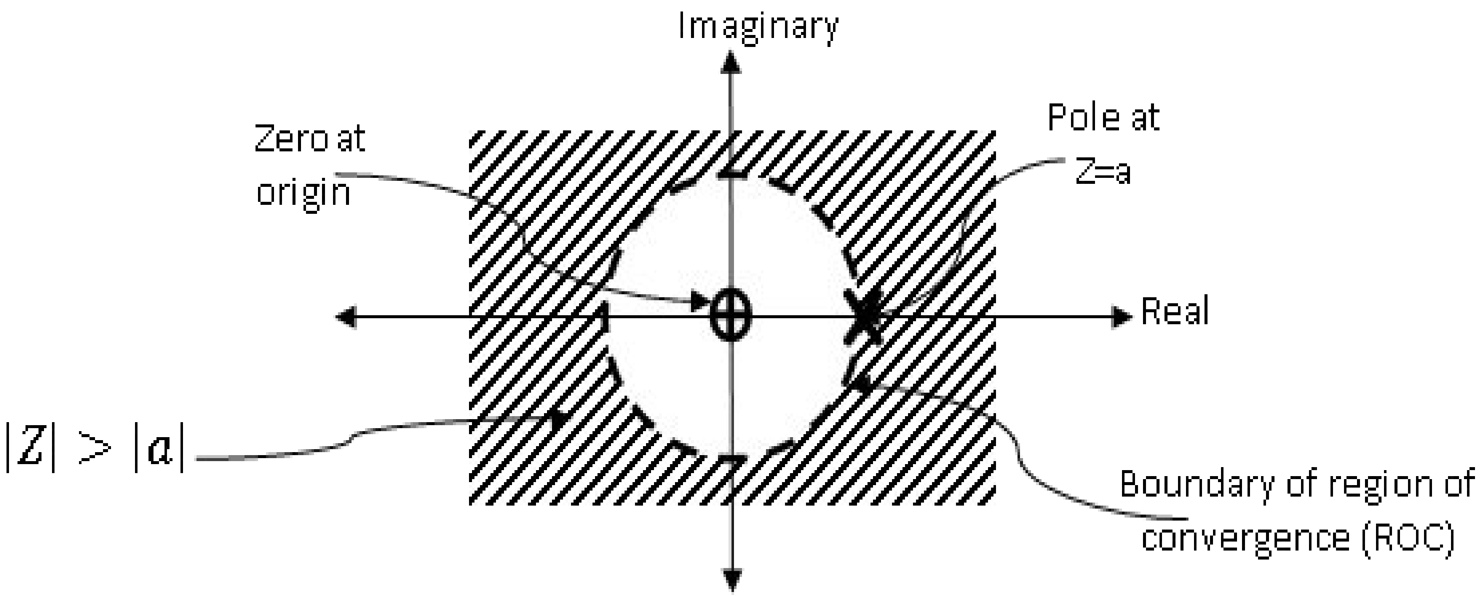
Most useful z-transforms can be expressed in the form, $X(z) = \frac{P(z)}{Q(z)}$, where P(z) and Q(z) are polynomials in z represent the numerator and denominator of X(z) respectively. The values of z for which P(z) and X(z) equal to zero are called the zeros of X(z), and the values with Q(z) = 0 and X(z) goes to infinity are called the poles.

$$\therefore \mathbf{Z}(a^n\ \mathbf{u(n)}) = \frac{z}{z-a}$$

ROC: $|z| > |a|$

Poles: z=a

Zeros: z=0



Example 2: Find the Z-transform including the region of convergence of

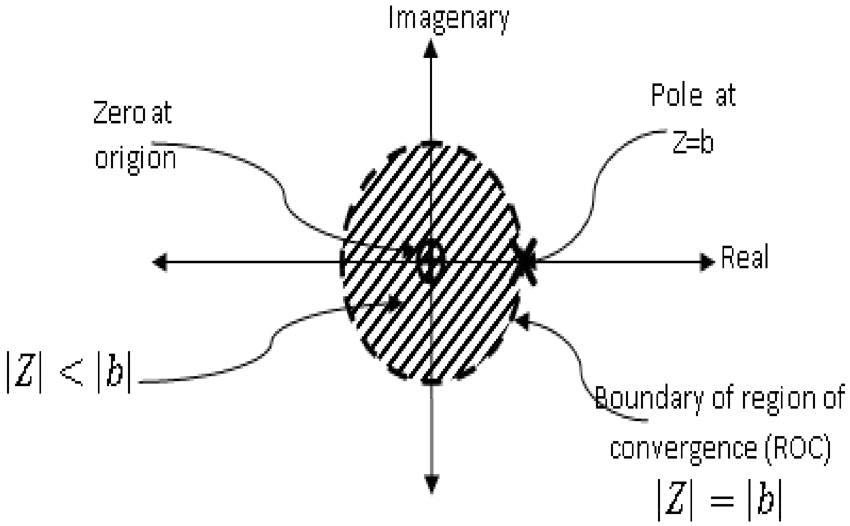
$$x(n) = -b^n u(-n-1)$$ Negative region

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \{-b^n u(-n-1)\} z^{-n} = -\sum_{n=-\infty}^{-1} (bz^{-1})^n$$

By letting n=−m in the above summation, that is changed the variables, and since interchanging the order of summation has not changed the sign, X(z) becomes

$$X(z) = -\sum_{m=1}^{\infty} \left(\frac{z}{b}\right)^m = 1-\sum_{m=0}^{\infty} \left(\frac{z}{b}\right)^m = 1-\frac{1}{1-\frac{z}{b}} = \frac{z}{z-b}$$



ROC of this transform is $|z/b| < 1 \rightarrow |Z| < |b|$

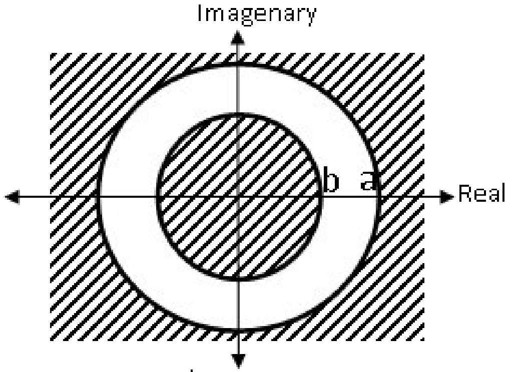
Example 3: Find the Z-transform including the region of convergence of $x(n) = a^n u(n) - b^n u(-n-1)$

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \{a^n u(n) - b^n u(-n-1)\} z^{-n} = \sum_{n=-\infty}^{\infty} \{a^n u(n)\} z^{-n} - \sum_{n=-\infty}^{\infty} \{b^n u(-n-1)\} z^{-n} = \frac{z}{z-a} + \frac{z}{z-b}$$

From previous two examples the regions of convergence of them are $|z| > |a|$ and $|z| < |b|$ respectively. And the final region of convergence of the result in this example is the intersection between these two regions.

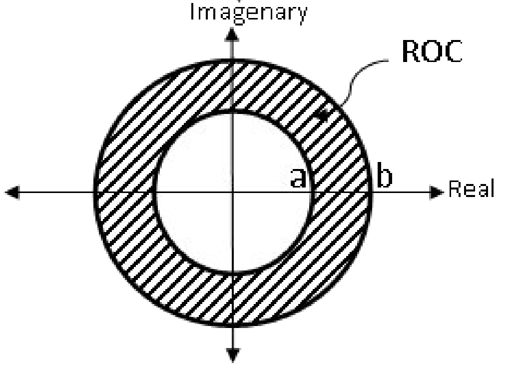
$$ROC\ is\ \{|z| > |a|\} \cap \{|z| < |b|\}$$



For drawing this region, it depends upon the values of the two variables a and b.

- If $|b| < |a|$, the above intersection is the empty set, i.e. the transform doesn't converge.

$$ROC\ is\ \{|z| > |a|\} \cap \{|z| < |b|\} = \textbf{null}$$



- If $|b| > |a|$, the transform converges in the annular region as shown in Figure below.

$$ROC\ is\ \{|z| > |a|\} \cap \{|z| < |b|\} = a < |z| < b$$