Example

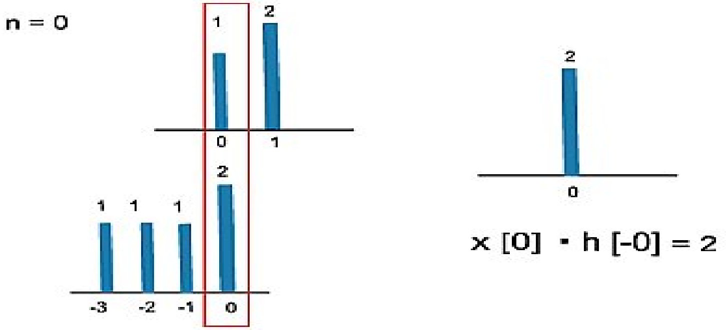
Linear Convolution of {x(n)} ={1, 2, 3, 1} and {h(n)} = {4, 3, 2, 1}

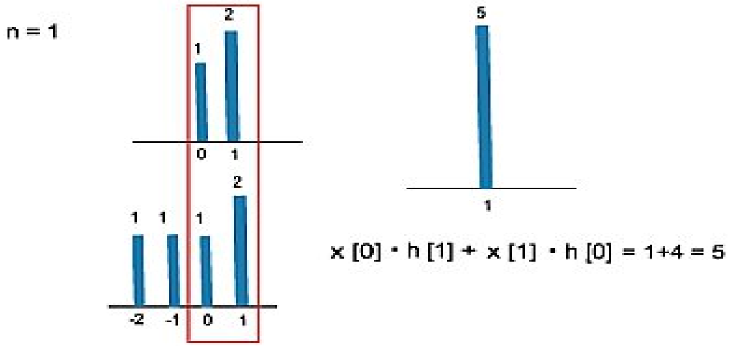
$$\mathbf{y(n)} = \sum_{k=0}^{n} x(k)\, h(n-k)$$

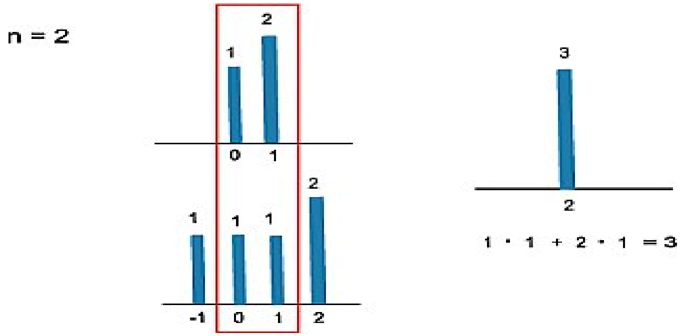
n = 0	$y(0) = \sum_{k=0}^{0} x(k)\, h(0-k)$	= x(0) h(0) = 1 . 4 = 4
n = 1	$y(1) = \sum_{k=0}^{1} x(k)\, h(1-k)$	= x(0) h(1) + x(1) h(0) = 1 . 3 + 2 . 4 = 11
n = 2	$y(2) = \sum_{k=0}^{2} x(k)\, h(2-k)$	= x(0) h(2) + x(1) h(1) + x(2) h(0) = 1 . 2 + 2 . 3 + 3 . 4 = 20
n = 3	$y(3) = \sum_{k=0}^{3} x(k)\, h(3-k)$	= x(0) h(3) + x(1) h(2) + x(2) h(1) + x(3) h(0) = 1 . 1 + 2 . 2 + 3 . 3 + 1 . 4 = 18
n = 4	$y(4) = \sum_{k=0}^{4} x(k)\, h(4-k)$	= x(0) ***h(4)*** + x(1) h(3) + x(2) h(2) + x(3) h(1) + ***x(4)*** h(0) = 1 . 0 + 2 . 1 + 3 . 2 + 1 . 3 + 0 . 4 = 11
n = 5	$y(5) = \sum_{k=0}^{5} x(k)\, h(5-k)$	= ***x(0) h(5) + x(1) h(4)*** + x(2) h(3) + x(3) h(2) + ***x(4) h(1)*** ***+ x(5) h(0)*** = 3 . 1 + 1 . 2 = 5
n = 6	$y(6) = \sum_{k=0}^{6} x(k)\, h(6-k)$	= ***x(0) h(6) + x(1) h(5) + x(2) h(4)*** + x(3) h(3) + ***x(4) h(2)*** ***+ x(5) h(1) + x(6) h(0)*** = 1 . 1 = 1
n = 7	$y(7) = \sum_{k=0}^{7} x(k)\, h(7-k)$	= ***x(0) h(7) + x(1) h(6) + x(2) h(5) + x(3) h(4) + x(4) h(3)*** ***+ x(5) h(2) + x(6) h(1) + x(7) h(0)*** = 0
y(n) = 0 for n < 0 and n > 6		↑Product terms in ***bold italics*** are zero.

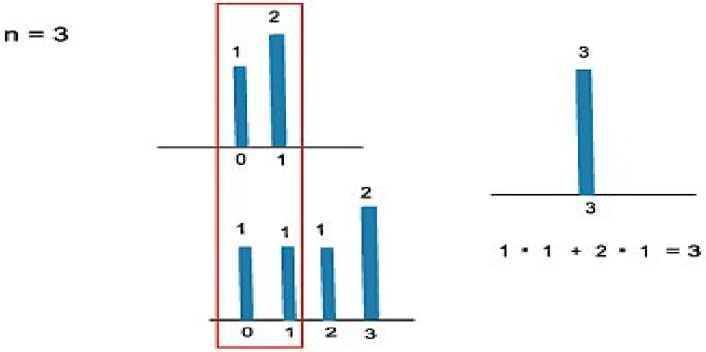
Example find the output response ***y[n] = x[n] * h[n] where x[n]={1,2} and h[n]={2,1,1,1}***

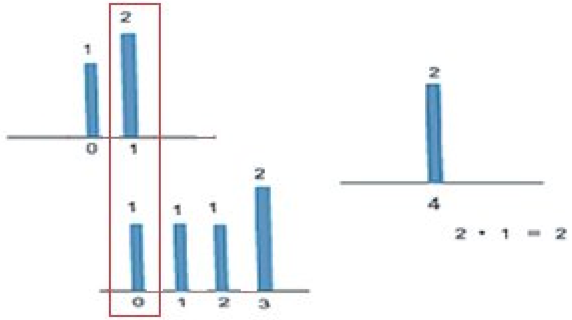
Solution: do time-reversing, and sliding across



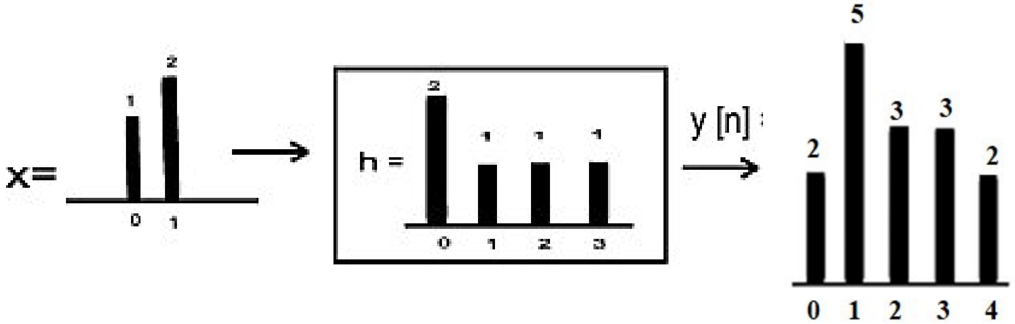








y[n] ={2, 5, 3, 3, 2}



This method can be used when both ***x[n]*** and ***h[n]*** have finite elements or duration. The length of ***y(n)***, equals the sum of the lengths of ***x(n)*** and ***h(n) minus 1***. Also the position of zero (n=0)at ***y(n)*** is equals the sum of the positions of x(n) and h(n) minus 1 as below:

Length of the output L_y **=** L_x **+** L_h **- 1**

Location of zero in output $\mathbf{0}_y$ **=** $\mathbf{0}_x$ **+** $\mathbf{0}_h$ **- 1**

Example : Given the input {x(n)} ={2, 1, -1, 0,-3} and impulse response {h(n)} = {1, 2, -1}

Find the output response y(n) = x(n) * h(n).

Solution:

$L_x=4$; $L_h=3$; $0_x=1$; $0_h=2$;

$=L_x + L_h -1 = 6$ length of y(n)

Location of y(n=0) $0_y=0_x+0_h-1=2$

h(n) \ x(n)	2	1	-1	0	-3
1	**2**	**1**	**-1**	**0**	**-3**
2	**4**	**2**	**-2**	**0**	**-6**
-1	**-2**	**-1**	**1**	**0**	**3**

y[n]= [2, 5,-1, -3, -2, -6, 3]

LTI systems can be characterized completely by their impulse response. The properties can also be characterized by their impulse response.

A. The Commutative Property of LTI Systems

A property of convolution in both continuous and discrete time is a ***Commutative Operation***. That is

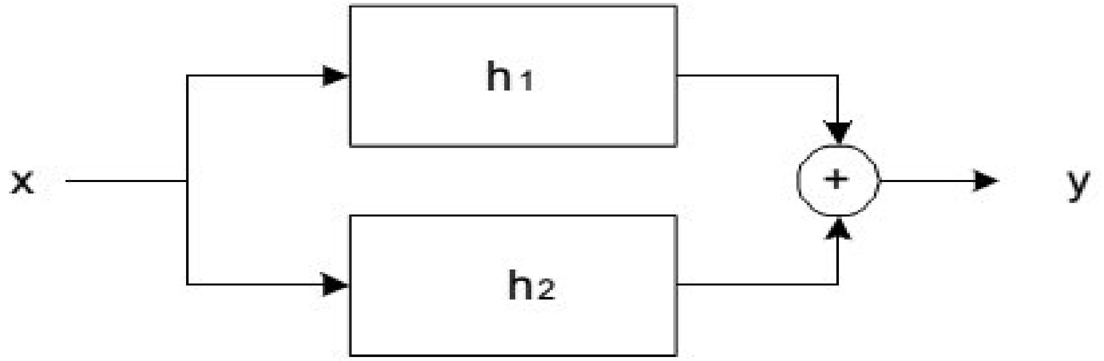
$$x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}h[k]x[n-k],$$

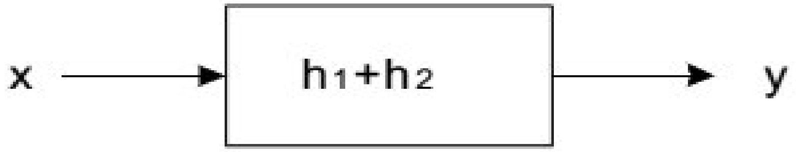
$$x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau\,.$$

B. The Distributive Property of LTI Systems $x*(h_1+h_2)=x*h_1+x*h_2$

For both discrete-time and continuous-time systems. The property means that summing the outputs of two systems is equivalent to a system with an impulse response equal to the sum of the impulse response of the two individual systems, as shown in the figure below.

The distributive property of convolution can be exploited to break a complicated convolution into several simpler ones.





For example, an LTI system has an impulse response $h[n]=u[n]$, with an input $x[n]=\left(\frac{1}{2}\right)^n u[n]+2^n u[-n]$. Since the sequence $x[n]$ is nonzero along the entire time axis. Direct evaluation of such a convolution is somewhat tedious. Instead, we may use the distributive property to express $y[n]$ as the sum of the results of two simpler convolution problems. That is,

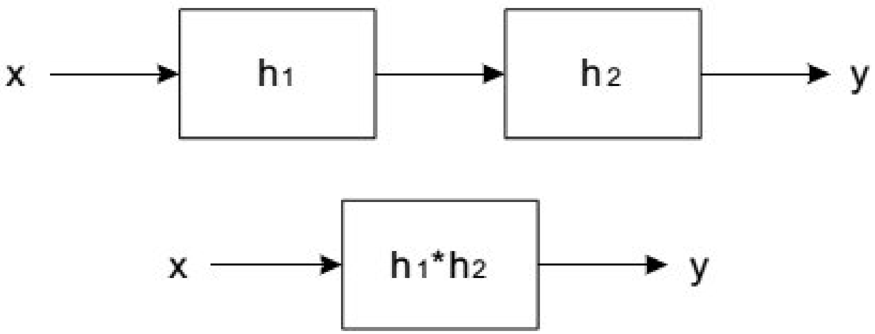
$x_1[n]=\left(\frac{1}{2}\right)^n u[n]$, $x_2[n]=2^n u[-n]$, using the distributive property we have

$$y[n]=\left(x_1(t)+x_2(t)\right)*h(t)=x_1(t)*h(t)+x_2(t)*h(t)=y_1[n]+y_2[n]$$

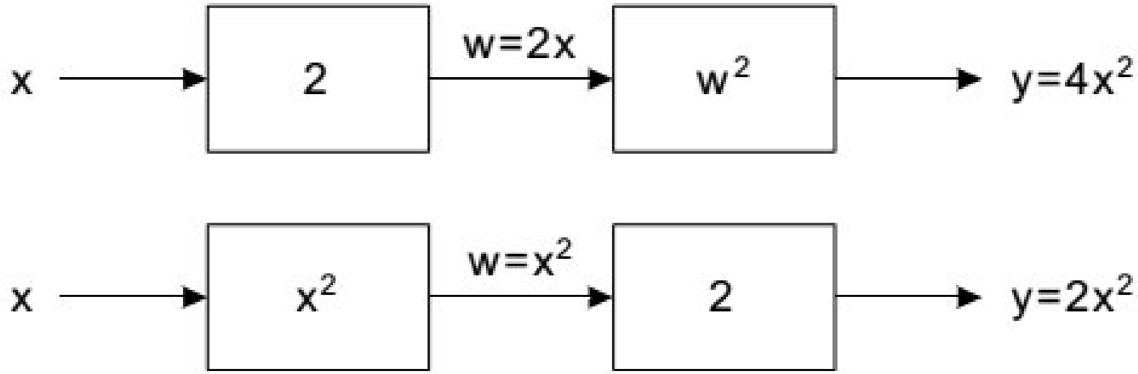
C. The Associative Property

For both discrete-time and continuous-time systems, the change of order of the cascaded systems will not affect the response.

$$x * (h_1 * h_2) = (x * h_1) * h_2$$



For nonlinear systems, the order of cascaded systems in general cannot be changed. For example, a two memoryless systems, one being multiplication by 2 and the other squaring the input, the outputs are different if the order is changed, as shown in the figure below.



D. LTI Systems with or without Memory:

A system is memoryless if its output at any time depends only on the value of its input at the same time. This is true for a discrete-time system, if $h[n]=0$ for $n \neq 0$.

$$y(t)=h(t)*x(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[n] = \ldots h[-2]\, x[n+2] + h[-1]\, x[n+1] + h[0]\, x[n] + h[1]\, x[n-1] + h[2]\, x[n-2] \ldots$$

In this case, the impulse response has the form

$$h[n]=K\delta[n],$$

where $K=h[0]$ is a constant and the convolution sum reduces to the relation

$$y[n]=Kx[n].$$

Otherwise the LTI system has memory.

For continuous-time systems, we have the similar results if it is memoryless:

$$h(t)=K\delta(t), \qquad y(t)=Kx(t).$$

Example: The LTI system with impulse response ***h(t)***= $-\,5\;\delta(t)$ is memoryless (static).

Example: LTI system with ***h(n)***=$4\;\delta(n-2)$ has memory because ***h(t)*** is not on the form $K\delta(t)$

E. Causality for LTI systems

A system is causal if its output depends only on the past and present values of the input signal. Specifically, for a discrete-time LTI system, this requirement is y[n] should not depend on x[k] for k > n. Based on the convolution sum equation, all the coefficients h[n - k] that multiply values of x[k] for k > n must be zero, which means that the impulse response of a causal discrete-time LTI system should satisfy the condition

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

$$y[n] = \ldots h[-2]\, x[n+2] + h[-1]\, x[n+1] + h[0]\, x[n] + h[1]\, x[n-1] + h[2]\, x[n-2] \ldots$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$h[n] = 0 \qquad \text{for } n < 0$$

A causal system is causal if its impulse response is zero for negative time; this makes sense as the system should not have a response before impulse is applied. A similar conclusion can be arrived for continuous-time LTI systems, namely

$$h(t) = 0 \qquad\qquad \text{for } t < 0$$

To proof the condition of convolution-based LTI system causality, let consider an LTI system having an output at time n_0, we can write the output of the system as summation of two sets as

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$y[n_0] = \sum_{k=-\infty}^{\infty} h[k]x[n_0-k] = \sum_{k=0}^{\infty} h[k]x[n_0-k] + \sum_{k=-\infty}^{-1} h[k]x[n_0-k] =$$
$$= \left[h[0]x[n_0] + h[1]x[n_0-1] + h[2]x[n_0-2] + \ldots\right] +$$
$$+ \left[h[-1]x[n_0+1] + h[-2]x[n_0+2] + h[-3]x[n_0+3]\ldots\right]$$

$\left[h[0]x[n_0] + h[1]x[n_0-1] + h[2]x[n_0-2] + \ldots\right]$-this set is the **present and the past** values of the input signal.
$\left[h[-1]x[n_0+1] + h[-2]x[n_0+2] + h[-3]x[n_0+3]\ldots\right]$-this set is the **future values** of the input signal.

Know, if the output of a causal system depends only on present and past-values of the input, then clearly the impulse response must satisfy the condition ***h[n] = 0* for *n* < 0**

Example: Check if LTI system whose impulse response given by ***h(t)*=** $e^{t-1}\,u(t)$ is causal or not

Solution: As unit step is defined as $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

Then the impulse response can be written as

***h(t)*=** $\begin{bmatrix} o & for\ t < 0 \\ e^{t-1} & for\ t > 0 \end{bmatrix}$

Hence this LTI system is causal as the condition above is satisfied.

F. Stability for LTI Systems

Recall that a system is *stable* if every bounded input produces a bounded output. For LTI system, if the input $x[n]$ is bounded in magnitude

$$|x[n]| \le B \qquad \textbf{for all } n$$

If this input signal is applied to an LTI system with unit impulse response $h[n]$, the magnitude of the output

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \right| \le \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]| \le B \sum_{k=-\infty}^{+\infty} |h[k]|$$

$y[n]$ is bounded in magnitude, and hence is stable if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty .$$

So discrete-time LTI system is stable if above Eq. is satisfied. The similar analysis applies to continuous-time LTI systems, for which the stability is equivalent to

$$\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty .$$

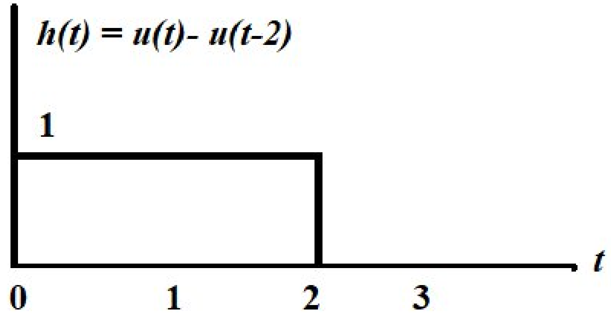
Example: Check if LTI system whose impulse response given by ***h(t)=*** $u(t) - u(t-2)$ is stable or not?

Solution: we have to check if $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$. or not.

$$\int_{-\infty}^{\infty} h(t) = \int_{-\infty}^{\infty} u(t) - u(t-2)\ dt$$

$$= \int_{-\infty}^{0} 0\ dt + \int_{0}^{2} 1\ dt + \int_{2}^{\infty} 0\ dt) = 2 < \infty$$

$\therefore$ **Stable LTI system.**



Example: consider a system that is pure time shift in either continuous time or discrete time.

In discrete time, $\sum_{k=-\infty}^{+\infty} |h[k]| = \sum_{k=-\infty}^{+\infty} |\delta[n-n_0]| = 1$,

while in continuous time, $\int_{-\infty}^{+\infty} |h(\tau)| d\tau = \int_{-\infty}^{+\infty} |\delta(t-t_0)| d\tau = 1$,

and we conclude that both of these systems are stable.

Example: The accumulator $h[n] = u[n]$ is unstable because $\sum_{k=-\infty}^{+\infty} |h[k]| = \sum_{k=0}^{+\infty} |u[n]| = \infty$.

Differential and Difference Equations

Differential equations are used to represent continuous time systems and difference equations are used to describe discrete time systems. These equations play a central role in describing the input-output relationships of a wide variety of electrical, mechanical and chemical systems.

Differential Equations (DE):-

A general CT system given by a linear constant coefficients differential equation (LCCDE) is described as

$$\sum_{k=0}^{N} a_k \frac{d^k\, y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k\, x(t)}{dt^k}$$

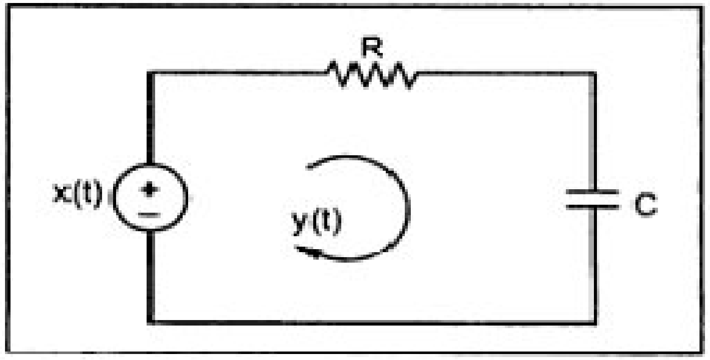
where ***x(t)*** is the input, ***y(t)*** is the output, and coefficients a_k , and b_k, are real constants. The order *N* refers to the highest derivative of *y(t)*. Recall the general solution to such a differential equation consists of 2 parts: **homogeneous and particular** for an input *x(t)* is given by

$$y(t) = y_p(t) + y_h(t)$$

where $y_p(t)$ is a particular solution (or **Forced Response** of the system) satisfying above Eq. and $y_h(t)$ is a homogeneous solution (or **complementary or the Natural Response**) satisfying the homogeneous differential equation below

$$\sum_{k=0}^{N} a_k \frac{d^k\, y(t)}{dt^k} = 0$$

Example : Write the differential equation for the RC circuit considered in figure below



Solution: In this circuit ***x(t)*** is the voltage input and ***y(t)*** is the current through the loop. It can be treated as output. By Kirchhoff's current low:

$$R\,y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)\, d\tau = x(t)$$

Differentiating above equation with respect to ***t*,**

$$R\,\frac{dy(t)}{dt} + \frac{1}{C}y(t) = \frac{dx(t)}{dt}$$

This is the differential equation of the RC circuit, comparing with the constant coefficient differential equation, we find that,

$$a_0 = \frac{1}{C},\ a_1 = R \text{ and } \quad b_0 = 0, b_1 = 1$$

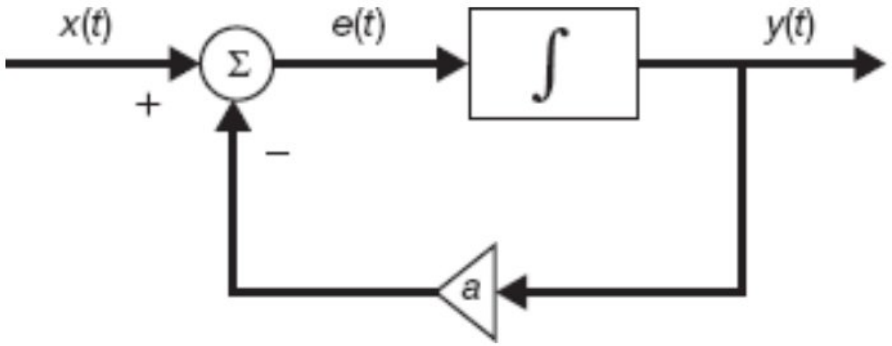
$$a_N\frac{d^N}{dt^N}y(t)+a_{N-1}\frac{d^{N-1}}{dt^{N-1}}y(t)+\cdots+a_1\frac{d}{dt}y(t)+a_0y(t)$$

$$= b_M\frac{d^M}{dt^M}x(t)+b_{M-1}\frac{d^{M-1}}{dt^{M-1}}x(t)+\cdots+b_1\frac{d}{dt}x(t)+b_0x(t)$$

$$\Rightarrow \sum_{k=0}^{N} a_k\frac{d^k}{dt^k}y(t) = \sum_{k=0}^{M} b_k\frac{d^k}{dt^k}x(t)$$

The differential equation can be solved to obtain the relationship between the input and output. Solutions of these differential equations require initial voltage across capacitor in RC circuit in above example. These are called the **initial conditions** which are normally specified with differential equations.

Example: The continuous-time system shown in Fig. below consists of one integrator and one scalar multiplier. Write a differential equation that relates the output *y(t)* and the input *x(t)*.



Solution:

Let the input of the integrator shown in Fig. above be denoted by ***e(t)***.
Then the input-output relation of the integrator is given by

$$y(t) = \int_{-\infty}^{t} e(\tau)\, d\tau$$

Differentiating both sides of above Eq. with respect to ***t***, we obtain

$$\frac{dy(t)}{dt} = e(t) \ldots\ldots(1)$$

Next, from Fig. the input ***e(t)*** to the integrator is given by $e(t) = x(t) - a\,y(t) \ldots\ldots(2)$

Substituting (2) in (1), yields:-

$$\frac{dy(t)}{dt} = x(t) - ay(t)$$

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

which is the required first-order linear differential equation

Exercise : Repeat above example for fig shown beside