Linear and nonlinear system

Also, the linear CT system must satisfy superposition condition (additive and homogeneity) . That is :

$$\boxed{\begin{aligned} \text{If } S[x_1(t)] &= y_1(t) \quad \text{and } S[x_2(t)] = y_2(t) \\ \text{then } S[ax_1(t) + bx_2(t)] &= ay_1(t) + by_2(t) \end{aligned}}$$

Example: check the linearity of the following system y(t) =x(t)?

Solution

Let x (t) = $x_1(t) + x_2(t)$ **then** $y(t) = a_1\, y_1(t) + a_2\, y_2(t)$.

$y_1(t) = x_1(t)$

$y_2(t) = x_2(t)$

Then the output y(t) corresponding to the input x(t) is

$y(t) = [\, a_1\, x_1(t) + a_2\, x_2(t)] = a_1\, y_1(t) + a_2\, y_2(t)$.

Linear system.

Example/ is $\mathbf{y(t) = 5t}\ \boldsymbol{x(t)}$ is linear or not?

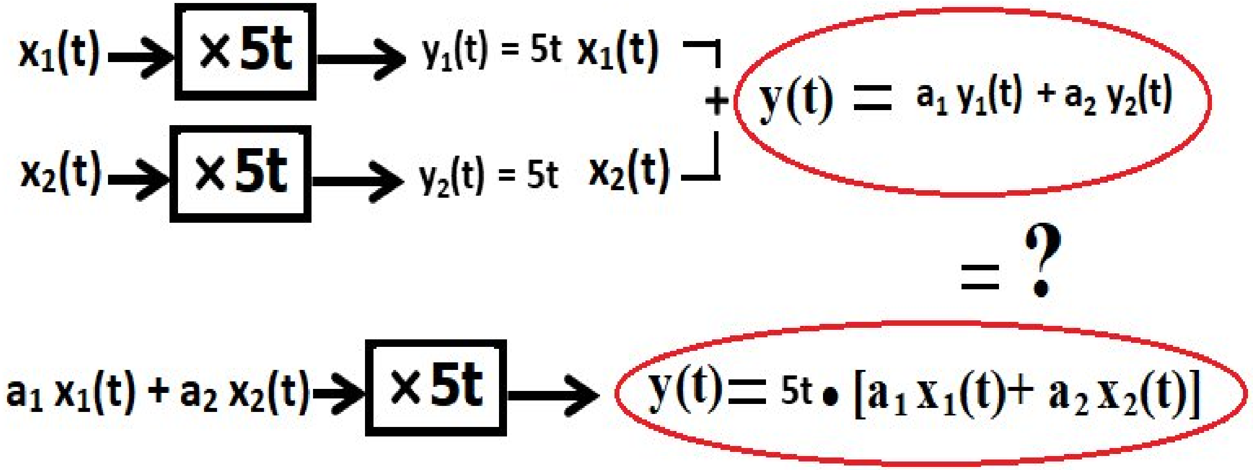
$y_1(t) = 5t\, x_1(t)$

$y_2(t) = 5t\, x_2(t)$ .... Scaling them to get

$y(t) = a_1\, y_1(t) + a_2 y_2(t) = a_1\, 5t\, x_1(t) + a_2\, 5t\, x_2(t)$ 1

$y(t) = 5t\, [a_1\, x_1(t) + a_2\, x_2(t)]$ 2

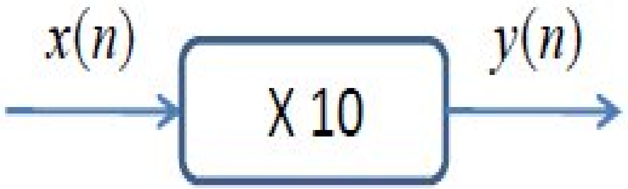
∴ eq. 1 = eq. 2 ... linear system



Note that the resistor and the capacitor mentioned before are linear systems.

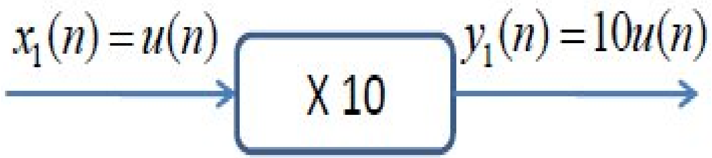
Example

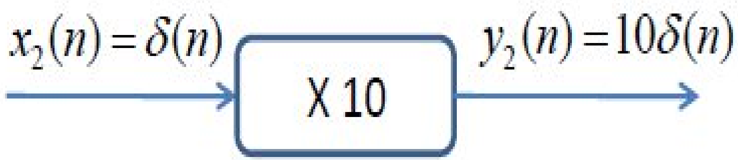
Let a digital amplifier, $y(n) = 10x(n)$



If the inputs are: $x_1(n) = u(n)$ and $x_2(n) = \delta(n)$

Outputs will be: $y_1(n) = 10u(n)$ and $y_2(n) = 10\delta(n)$, respectively.





If we apply combined input to the system: $x(n) = 2x_1(n) + 4x_2(n) = 2u(n) + 4\delta(n)$

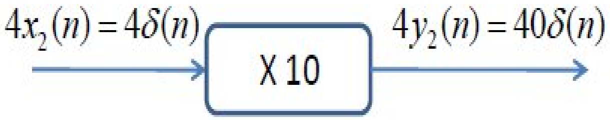
The output will be:

$$y(n) = 10x(n) = 10(2u(n) + 4\delta(n)) \boxed{= 20u(n) + 40\delta(n)}$$

$$\boxed{\text{Individual outputs: } \begin{array}{l} 2y_1(n) = 2 \times 10x_1(n) = 20u(n) \\ 4y_2(n) = 4 \times 10x_2(n) = 40\delta(n) \end{array}}$$

+

Linear System



Example

$$x(n) \rightarrow \boxed{\text{System}} \rightarrow y(n) = x^2(n)$$

$$x_1(n) = u(n) \rightarrow \boxed{\text{System}} \rightarrow y_1(n) = u^2(n) \qquad x_2(n) = \delta(n) \rightarrow \boxed{\text{System}} \rightarrow y_2(n) = \delta^2(n)$$

If the input is: $x(n) = 4x_1(n) + 2x_2(n)$

Then the output is:

$$\begin{aligned} y(n) &= x^2(n) = (4x_1(n) + 2x_2(n))^2 \\ &= (4u(n) + 2\delta(n))^2 = 16u^2(n) + 16u(n)\delta(n) + 4\delta^2(n) \\ &= 16u(n) + 20\delta(n). \end{aligned}$$

Individual outputs:

$$4y_1(n) = 4 \times x_1^2(n) = 4u(n)$$
$$2y_2(n) = 2 \times x_2^2(n) = 2\delta(n)$$

Non Linear System

Example: is $y(t) = x^2(t)$ linear or nonlinear system ?

Solution:

$y_1(t) = x_1^2(t)$

$y_2(t) = x_2^2(t)$....

The summation and scaling of these sub-o/p is

$y(t) = a_1 y_1(t) + a_2 y_2(t) = a_1 x_1(t)^2 + a_2 x_2(t)^2$ 1

then the output y(t) corresponding to the input $x(t) = [a_1 x_1(t) + a_2 x_2(t)]$ is

$y(t) = [a_1 x_1(t) + a_2 x_2(t)]^2 = a_1^2 x_1(t)^2 + a_2^2 x_2(t)^2 + 2 a_1 x_1(t) a_2 x_2(t)$ 2

Which is not equal to $a_1 y_1(t) + a_2 y_2(t)$. Hence the system is said to be **nonlinear**.

Example: is *y(t) = cos (x(t))* linear or nonlinear system ?

Solution

$y_1(t) = \cos(x_1(t))$, and $y_2(t) = \cos(x_2(t))$

Then the output y(t) corresponding to the input x(t) is

$y(t) = \cos(\alpha x_1(t) + \beta x_2(t))$

$= \cos(\alpha x_1(t)) \cos(\beta x_2(t)) - \sin(\alpha x_1(t)) \sin(\beta x_2(t))$

$\neq \alpha \cos(x_1(t)) + \beta \cos(x_2(t))$

The system is NOT linear.

A system is said to be time variant if its input and output characteristics vary with time. Otherwise, the system is considered as time invariant.

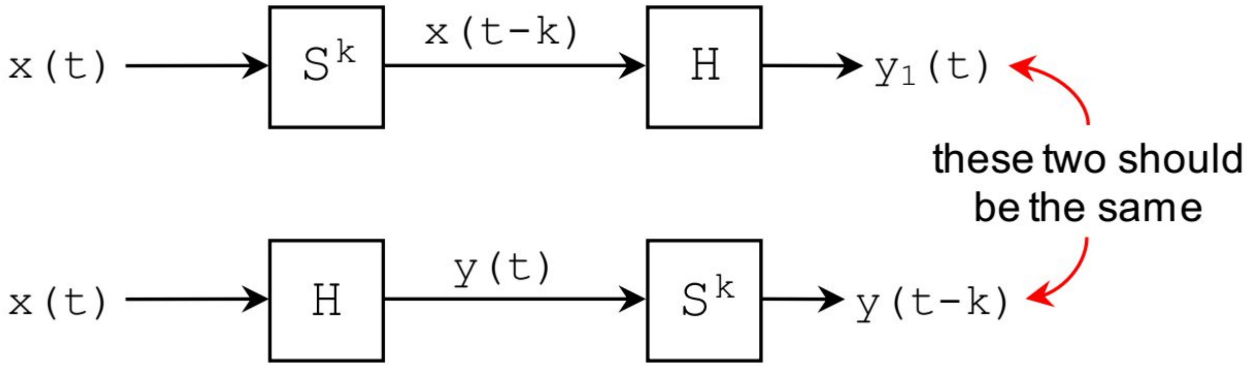
On another words, If the input $x(t - \tau)$ produces a response $y(t - \tau)$ where τ is any real constant, the system is called a ***time-invariant* system.** In other words, a system is called *time-invariant if a time shift (delay or advance) in the input signal causes the same time shift in the output signal*. Thus, for a continuous-time system, the system is time-invariant if

$$f(x(t-\tau)) = y(t-\tau)$$

For any real value of **τ**. For a discrete-time system, the system is time-invariant (or shift-invariant) if

$$f(x(n-k)) = y(n-k)$$

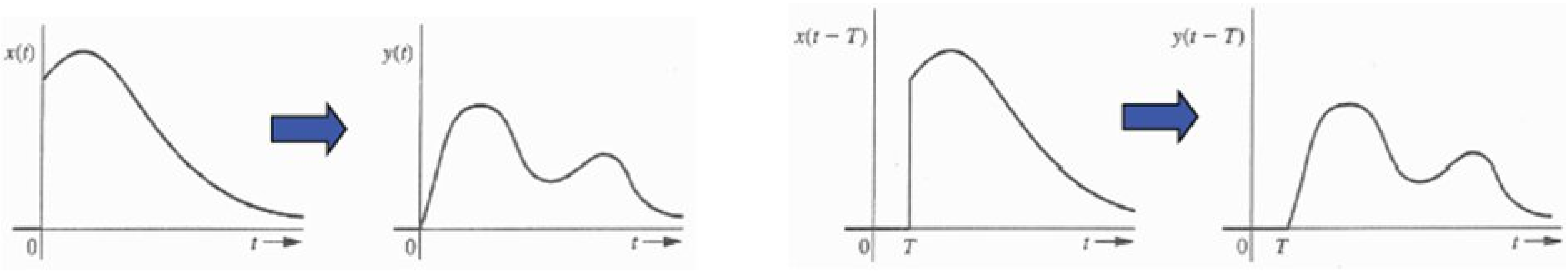
For any integer ***k*** **.** A system which does not satisfy above conditions is called a time-varying system.

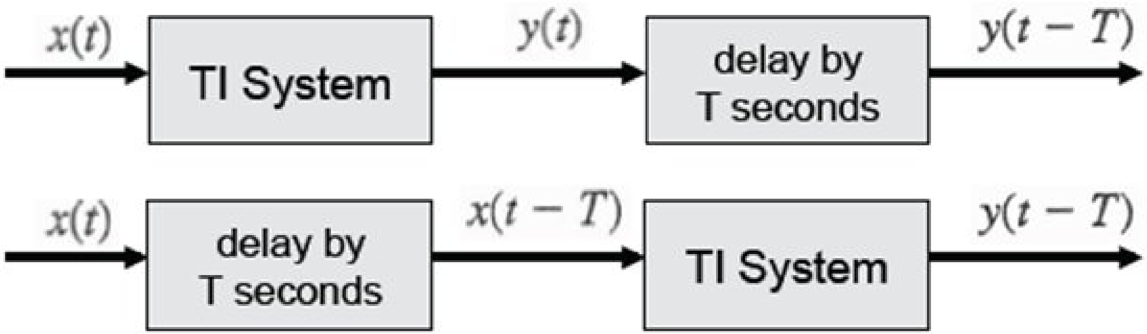


We have a system y(t) = H(x(t)). The response of the system to a shifted input x(t-k) should be the same as if the output y(t) has been shifted by k, i.e., y(t-k):

Time-Invariant and Time-Varying Systems

- **Time-invariant system** is one whose parameters do not change with time:





Example : The system $y[n]$=K where K is constant is time invariant because it does not depend on time.

Example: check the time invariance of the system $y(t) = x(-t)$

Solution : The output due to shifting the input is $y_1(t) = T[x(t-T)] = x(-t-T)$

The output due to shifting the equation (replace each t by (t-T) is

$y_2(t-T) = x(-(t-T)) = x(-t + T)$ ∴ $y_1 \neq y_2$. Hence, the system is time variant.

Example : is *$y(t) = x(t - 2) + x(2 - t)$* time variant or not?

Solution: Find output due to shifting in input which is

$x(t - to) \rightarrow y_1(t) = x(t - to - 2) + x(2 - t - to)$

Find $y_2(t - to)$ which corresponding to output shifting (replacing each *t* by *t-to*)

$y_2(t - to) = x(t - to - 2) + x(2 - t + to)$.

$y_1 \neq y_2$: Hence system is time-variant.

Example: Examine $y[n] = x[n] + n\, x[n+1]$ for time invariance.

Solution: Notice that the equation has a **time-varying coefficient**, *n*.

The output *y[n]* corresponding to *x[n]* is already given above. Delaying *y[n]* by *k* gives

$y_1(n-k) = x(n-k) + (n-k)\, x(n-k+1) \quad \rightarrow$ (A)

Compare with $y_2(n) = T[x(n-k)] = x(n-k) + n\, x(n-k+1) \rightarrow$ (B)

Since (A) ≠ (B), the system is time varying.

Note that in applying the time-invariance test, we time-shift the input signal only, not the coefficients.

Exercise: Which of the following system is time-invariant? (a) $y(t) = 3x(t)$ (b) $y(t) = t\, x(t)$

Linear Time-Invariant (LTI) Systems

In previous sections, a number of basic system properties are introduced. Two of these, linearity and time invariance, play a fundamental role in analysis and simulation of systems because of the many physical processes that can be modeled by linear time-invariant (**LTI**) systems. It will be shown that the input-output relationship for LTI systems is described in terms of a **convolution operation.** The importance of the convolution operation in LTI systems comes from the fact that knowledge of the response of an LTI system to the **unit impulse input** allows to find its output to any input signals. Specifying the input-output relationships for LTI systems by differential and difference equations will also be discussed.

Continuous LTI System

The ***impulse response*** completely characterizes the input-output behavior of LTI systems. Hence, properties of the system are related to the system's impulse response.

<u>The Impulse Response:</u>

The response to a **delta function** of an arbitrary linear system defined by an operator **T** is defined as

$$h(t) = \mathbf{T}\{\delta(t)\}$$

The function *h(t)* is called the *impulse response*. For LTI systems

$$h(t-\tau) = \mathbf{T}\{\delta(t-\tau)\}$$

The Convolution Integral :

A convolution is an integral that expresses the amount of overlap of one function as it is shifted over another function.

The response of a LTI system to an arbitrary input $x(t)$ is given by

$$\mathbf{y(t)} = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\mathrm{d}\tau$$

Above equation is referred to as the ***convolution integral*** or the ***superposition integral*** and indicates that a continuous-time LTI system is completely characterized by its impulse response *h(t)*. The convolution of the signal ***x*(*t*)** and the impulse response ***h*(*t*)** is denoted as

$$y(t) = x(t) * h(t)$$

Properties of the Convolution Integral:

The most important properties of the convolution are a consequence of the superposition properties of linear time-invariant systems.

1- The first basic property is that it is a commutative operation

$$x(t) * h(t) = h(t) * x(t)$$

2- A second useful property is that it is associative

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

3- The third is the distributive property

$$y(t) = x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

Convolution Integral Operation

Applying the commutative property of convolution to below equation

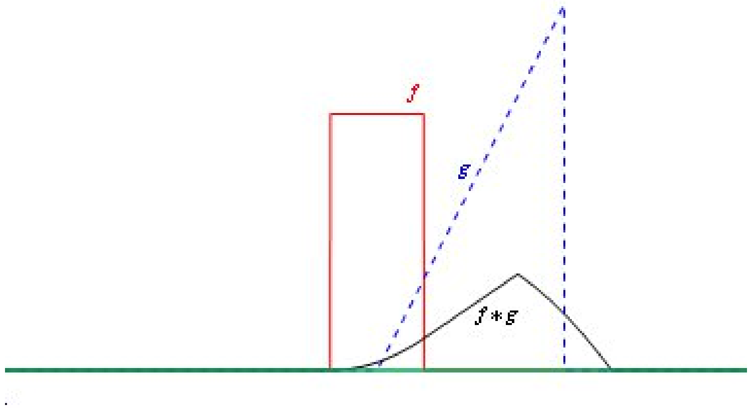
$$\mathbf{y(t)} = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\mathrm{d}\tau$$

We obtain

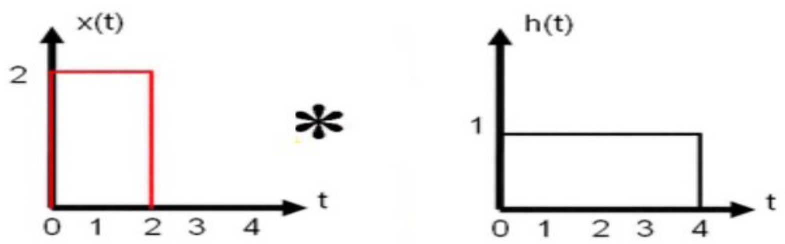
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau$$

The convolution integral operation involves the following four steps:

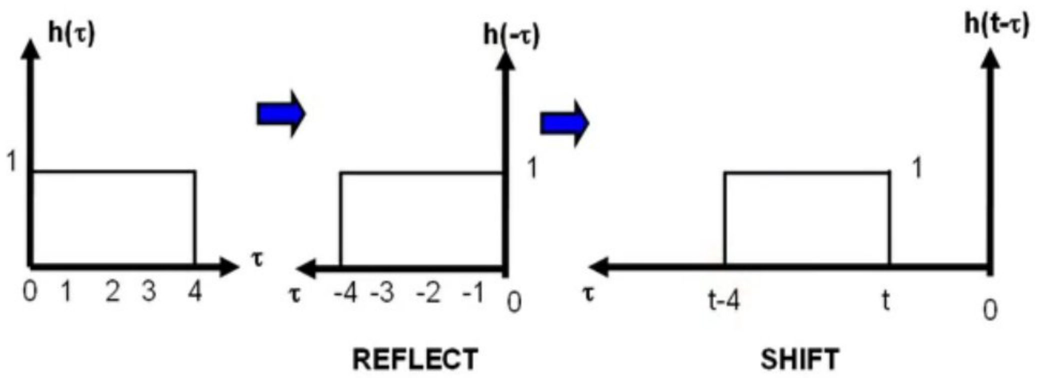
1. First, we time-reverse h(τ) and shift it to form h(t - τ). Note: the independent variable of h(t - τ) is τ, not t. The variable t is the shift parameter, i.e. the function h(τ) is shifted by an amount t.
2. Next, fix t and multiply x(τ) with h(t - τ) for all values of τ.
3. Then integrate x(τ) h(t - τ) over all τ to get y(t) which is a single value that depends on t. Remember that τ is the integration variable and that t is treated like a constant when doing the integral.
4. Finally, repeat for all values of t to produce the entire output *y(t)*.

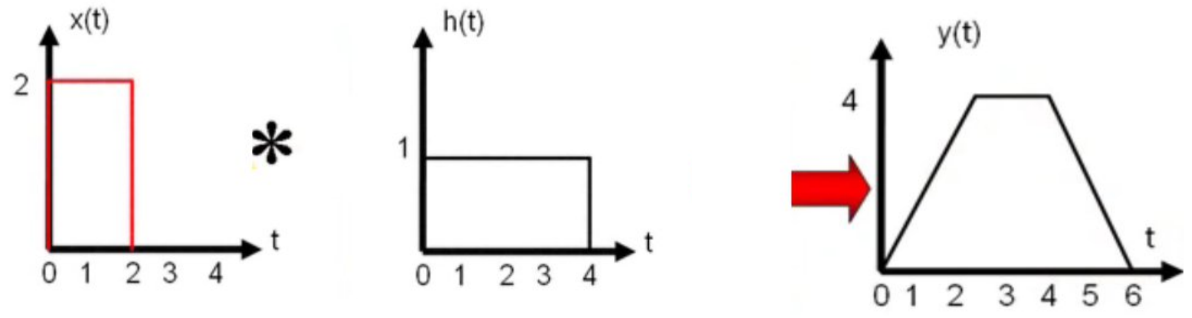


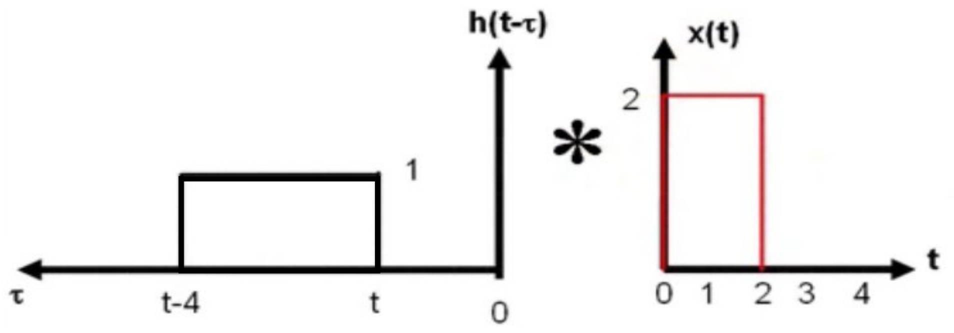
Examples : Evaluate $y(t) = x(t) * h(t)$, where $x(t)$ and $h(t)$ are shown below



Solution:





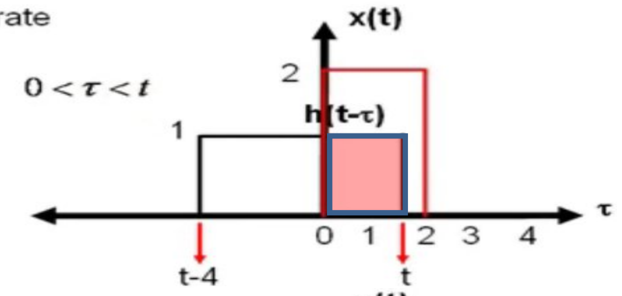


Multiply and Integrate

$$y(t) = \int_0^t 2 \cdot 1 d\tau \qquad 0 < \tau < t$$

$$y(t) = 2 \cdot (\tau|_0^t)$$

$$y(t) = 2t$$



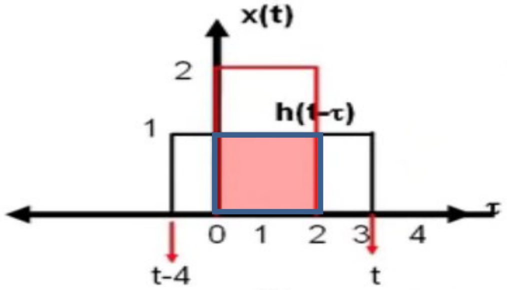
$t - 4 < 0 \qquad t > 2$

$2 < t < 4$

$0 < \tau < 2$

$$y(t) = \int_0^2 2 d\tau$$

$$y(t) = 4$$



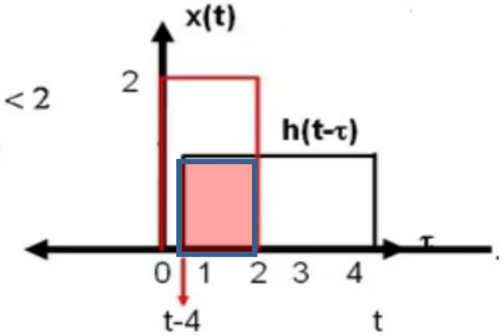
$$y(t) = \int_{t-4}^2 2 \cdot 1 d\tau \qquad t-4 < \tau < 2$$

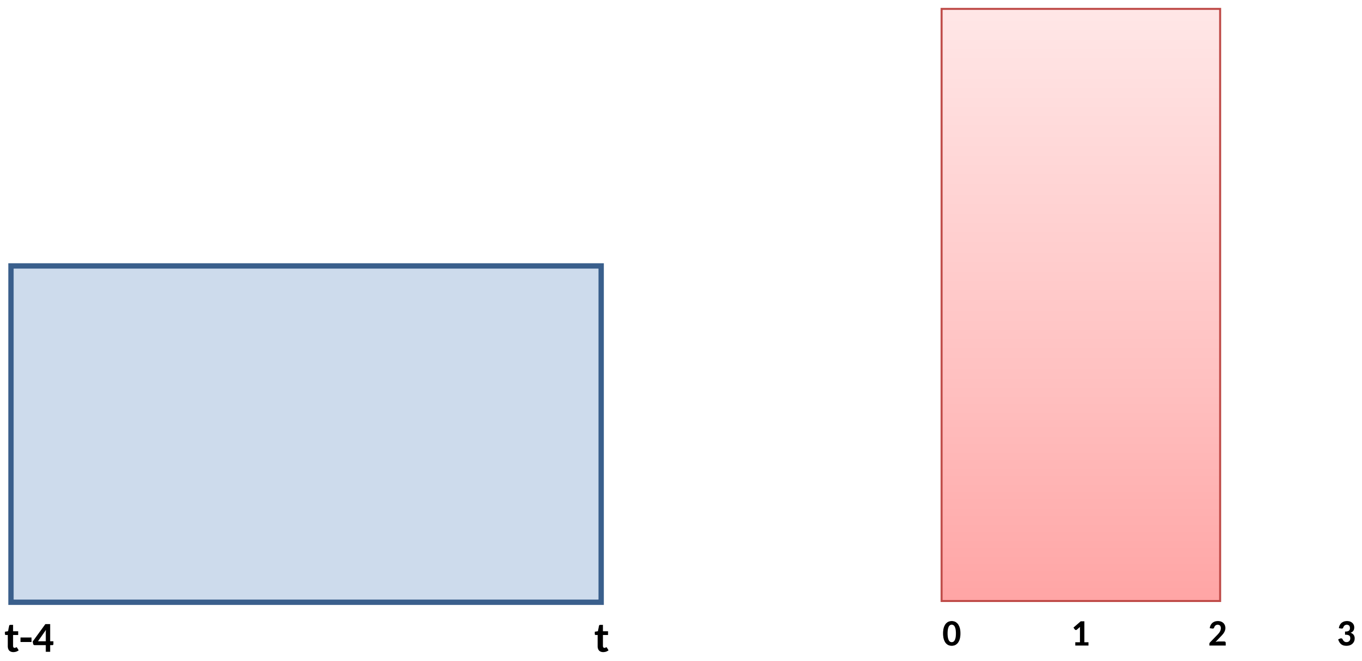
$t < 6$

$$y(t) = 2 \cdot (\tau|_{t-4}^2)$$

$$y(t) = 2(2 - (t-4))$$

$$y(t) = -2t + 12$$




t-4
t
0
1
2
3

The Impulse Response: The impulse response (or unit sample response) ***h[n]*** of a discrete-time LTI system (represented by **T**) is defined to be the response of the system when the input is ***δ[n],*** that is,

$$h[n] = T\{\delta[n]\}$$

The output response ***y [n]***of LTI system to an arbitrary input ***x [n]***can be expressed as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\ h[n-k]$$

The Convolution Sum:

The convolution sum of two sequences ***x [n]*** and ***h [n]*** is defined by

$$y[n] = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x[k]\ h[n-k]$$

Thus, again, we have the fundamental result that the output of any discrete-time LTI system is the convolution of the input ***x[n]*** with the impulse response ***h[n]***of the system.

Properties of the Convolution Sum

The following properties of the convolution sum are analogous to the convolution integral properties

1- Commutative: $x[n] * h[n] = h[n] * x[n]$

2- Associative: $\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$

3- Distributive: $x[n] * \{h_1[n]\} + h_2[n]\} = x[n] * \{h_1[n] + x[n] * h_2[n]\}$

Convolution Sum Operation:

Similar to the continuous-time case, the convolution sum operation involves the following four steps:

1. The impulse response $h[k]$ is time-reversed (that is, reflected about the origin) to obtain $h[-k]$ and then shifted by n to form $h[n-k] = h[-(k-n)]$ which is a function of k with parameter n.

2. Two sequences $x[k]$ and $h[n-k]$ are multiplied together for all values of k with n fixed at some value.

3. The product $x[k]\, h[n-k]$ is summed over all k to produce a single output sample $y[n]$.

4. Steps 1 to 3 are repeated as n varies over $-\infty$ to ∞ to produce the entire output $y[n]$.