Comparison of Digital and Analog Filters

Digital filter

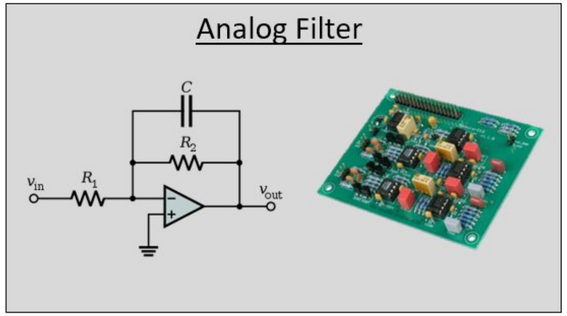
1. It operates on digital samples (or sampled signals
2. It is governed (or defined) by linear difference equations.
3. It consists of adders, multipliers, and delay elements implemented in digital logic (either in hardware or software or both).
4. In digital filters, the filter coefficients are designed to satisfy the desired frequency response

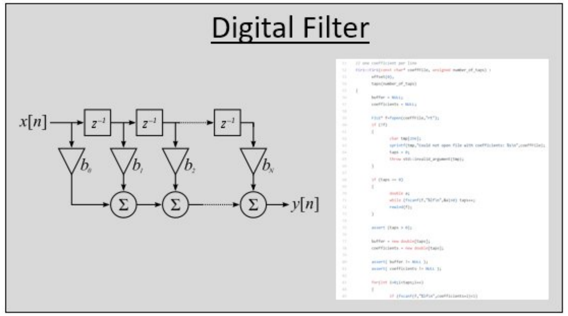
Analog filter

1. It operates on analog signals (or actual version) of the signal.
2. It is governed (or defined) by linear differential equations.
3. It consists of electrical components like resistors, capacitors, and inductors.
4. In analog filters, the approximation problem is solved to satisfy the desired frequency response. response

Digital filters Characteristics

1. Digital filter have truly linear phase response.
2. The performance of digital filter does not vary with environment
3. Several input signals or channels can be filtered by one digital filter without the need to change the hardware.
4. Both filtered and unfiltered data can be saved for further use.
5. High precision achieved by digital filter and depend on word length.
6. Digital filter can be used for low frequency.
7. *The signal bandwidth of i/p signal is limited by ADC and DAC.*
8. *The bandwidth of the digital filter is much lower than an analogue filter.*
9. *Quantization noise is present.*
10. *It is expensive.8*





Types of Digital Filters. IIR and FIR

Digital filters are broadly divided into two classes:-

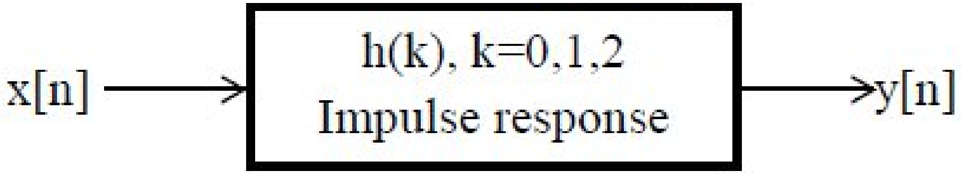
1- Infinite impulse response (IIR)

2- Finite impulse response (FIR)

The types of filters which make use of feedback connection to get the desired filter implementation are known as recursive filters. Their impulse response is of infinite duration. So they are called IIR filters. The type of filters which do not employ any kind of feedback connection are known as non-recursive filters. Their impulse response is of finite duration. So they are called FIR filters.

$$y[n] = \sum_{k=0}^{\infty} h(k)\, x[n-k]$$

For IIR filters.



$$y[n] = \sum_{k=0}^{N-1} h(k)\, x[n-k]$$

For FIR filters.

The choice between FIR and IIR filters depends on the relative advantages of the two filters

1) FIR has linear phase response and this is important for data transition, biomedicine, digital audio and image processing.

2) FIR filters realized no recursively are always stable

3) FIR requires more coefficients for sharp cut off than IIR.

4) Analogue filters can be readily transformed into equivalent IIR digital filters meeting similar specifications.

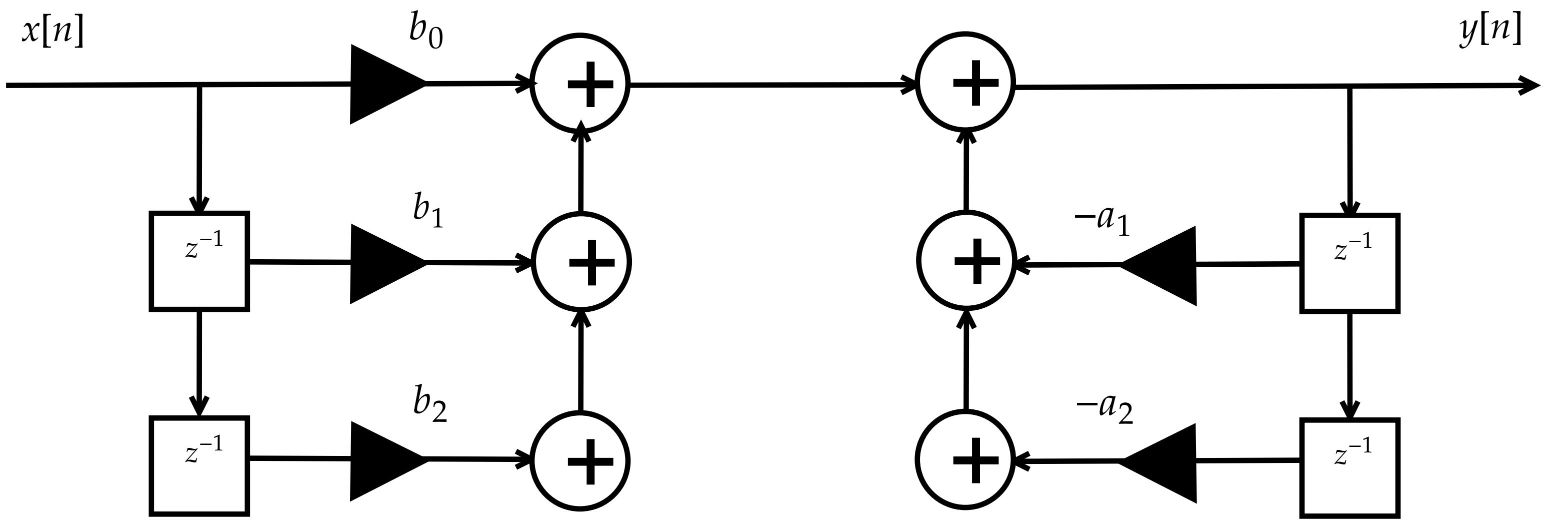
Recall our expression for a linear, constant-coefficient difference equation:

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \ldots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$

This equation can be written briefly using summations:

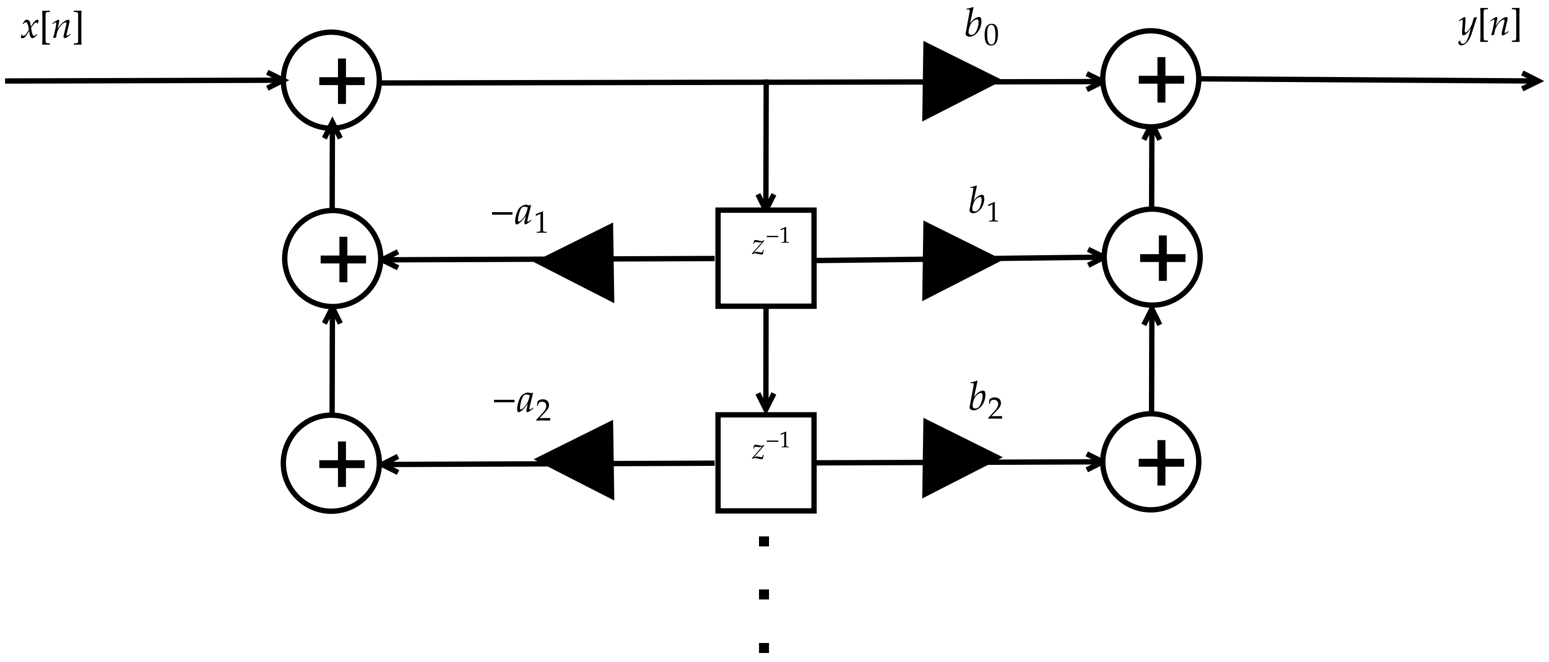
$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{l=0}^{M} b_l x[n-l]$$

We can draw a signal flow graph implementation of this equation:



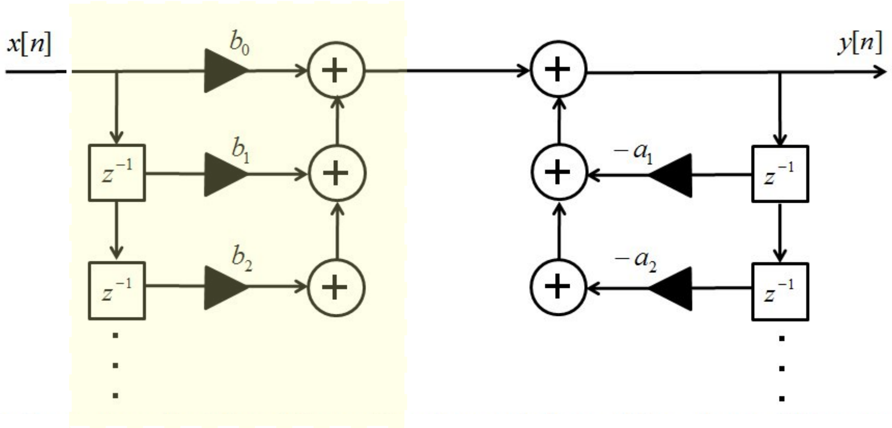
This is known as the **Direct Form I implementation** of the above difference equation.
Can we implement this more efficiently?

- One of the more elementary aspects of the field of digital signal processing is to develop more efficient implementations of digital filters, as well as improve their ability to produce accurate results with less numerical precision.
- A more efficient implementation of our filter is a **Direct Form II:**



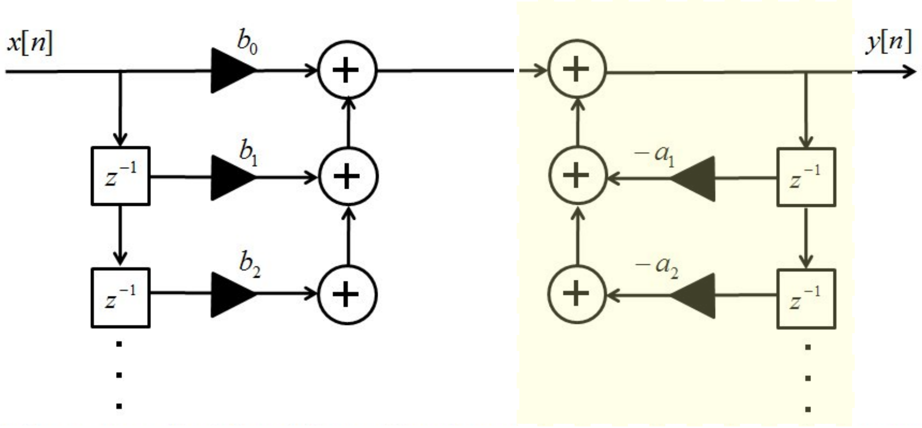
- This filter has the same transfer function but shares the delay element between the feedforward (moving average/finite impulse response) and feedback (autoregressive/infinite impulse response) portions of the filter.
- Analog differential equations can be represented by similar signal flow graphs, but their implementation involves physical components (e.g., RLCs, op amps).

- Consider a filter with only feedforward components:
- The transfer function is: $y[n] = \sum_{l=1}^{M} b_l x[n-l]$ $\boldsymbol{H(z) = \frac{Y(z)}{X(z)} = \sum_{l=0}^{M} b_l z^{-l}}$



- Since the impulse response of this filter, *h*[*n*], has a finite number of nonzero terms, this filter is referred to as a finite impulse response (FIR) filter. Observe that this filter only has zeroes.

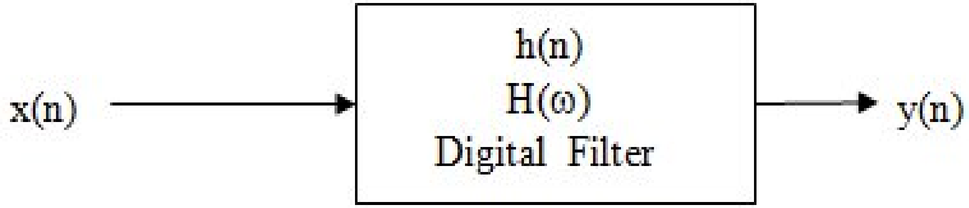
- Next, consider a filter with only feedback components:
- The transfer function is: $y[n] = -\sum_{k=1}^{N} a_k y[n-k] + b_0 x[n]$ $\boldsymbol{H(z) = \frac{b_0}{1+\sum_{k=1}^{N} a_k z^{-k}}}$



- This is an all-pole filter with an infinite impulse response (IIR). Why?

Digital Filter Design

A digital filter is a discrete time LTI system which can process the discrete time signal. There are various structures for the implementation of digital filters. The actual implementation of an LTI digital filter can be either in software or hardware form, depending on applications. A digital filter is in fact a linear causal DSP system with impulse response h(n) and frequency response $H(\omega) = |H(\omega)| \angle \varphi(\omega)$. Where the shape of $|H(\omega)|$ gives the type of the filter (lowpass, highpass,…..) (amplitude characteristics), and $\varphi(\omega)$ gives its phase characteristics . As mentioned before, depending on the type of h(n), these filters are also classified into FIR and IIR. The properties and design procedures for these two classes are different. In this course, only FIR filter design is given for its simplicity.



<u>FIR Digital Filter Design:</u>

FIR filters are always stable having no feedback and with linear phase characteristics This linear phase of the FIR filter is always preferred to avoid phase distortion at the output signal y(n). , i.e. $\varphi(\omega) = -k\omega$ (k is a constant and the –sign indicates the phase lagging of the filter).

w is called a digital frequency (in rad) since it is normalized to the sampling frequency f_s of the A/D converter, i.e.:

$\omega = 2\pi f/f_s$ where f is the actual analogue frequency(in Hz) of the continuous signal x(t). This linear phase of the FIR filter is always preferred to avoid phase distortion at the output signal y(n).

Linear phase condition:

The FIR filter with impulse response h(n)={h(0), h(1), h(2),….h(N-1)} with N elements has a linear phase if these elements are **symmetric** about a midpoint α **=(N-1)/2**, i.e., h(n)=h(N-1-n), where n=0,1,2,…N-1. This midpoint may be an integer if N odd or a fraction if N is even:

For N even, the frequency response is given by

$$|H(\omega)| = \sum_{n=0}^{\frac{N}{2}-1} 2h(n)\, cos\,(\omega(n-\alpha))$$

where α **=(N-1)/2** (the midpoint),

$\theta(\omega) = -\alpha\omega$ (linear phase)

For N Odd, , the frequency response is given by

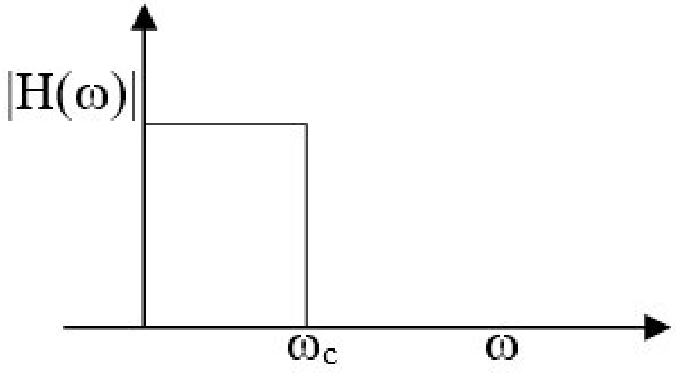
$$|H(\omega)| = h(\alpha) + \sum_{n=0}^{\alpha-1} 2h(n)\, cos\,(\omega(n-\alpha))$$

where α **=(N-1)/2** (the midpoint),

$\theta(\omega) = -\alpha\omega$ (linear phase)

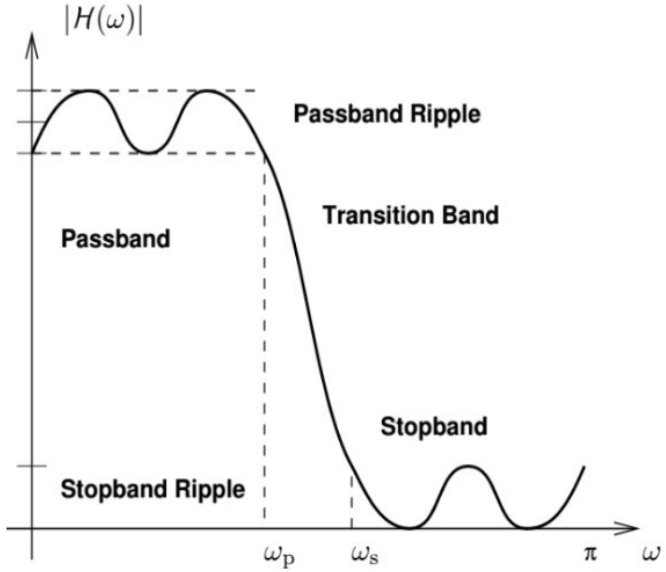
It should be noted that N odd is mostly used since the symmetry will be around a real midpoint since α is integer.

After studying linear phase condition and deriving the frequency response H(⟟) of the filter, We discuss how to find ***h(n)*** as symmetric around ✔ taking N as an **odd** number. **Theoretically, to have ideal lowpass filter** as shown then, we must use infinite number of elements (N is very large), but due to the finite length of N, there will be a roll of and sidelobes at stopband region. To reduce these sidelobes (SLL) in stopband, we use what is called windows . Several types of windows are used, but the most commonly used are listed below:



Window type	**K-factor**	**Stopband ripple(Sidelobe Level SLL)**
Rectangular	2	-21dB
Bartlet	4	-25dB
Hanning	4	-44dB
Hamming	4	-53dB
Blackman	6	-74dB

Where the K-factor is a parameter used in finding **N**.

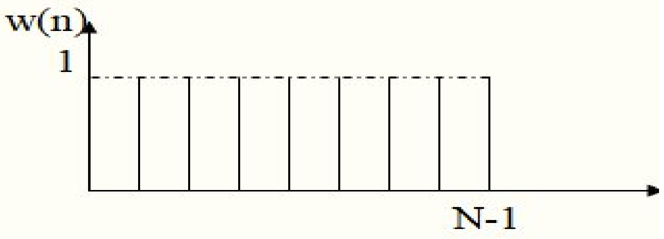


1- Rectangular:

w(n)={ 1 $\quad$ N-1≥ n≥ 0

{ 0 $\quad$ elsewhere

$\alpha=(N-1)/2$

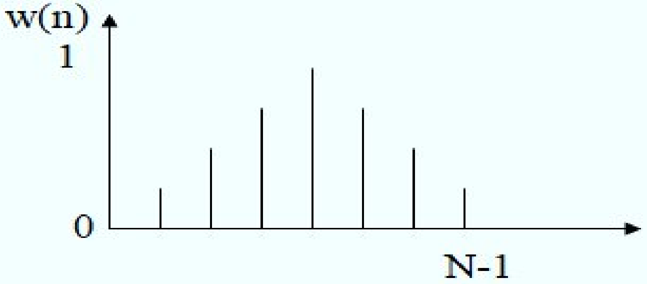


2-Bartlet(triangular):

w(n)={n/α $\quad$ α≥ n≥ 0

{2-(n/α) $\quad$ N-1≥ n> α

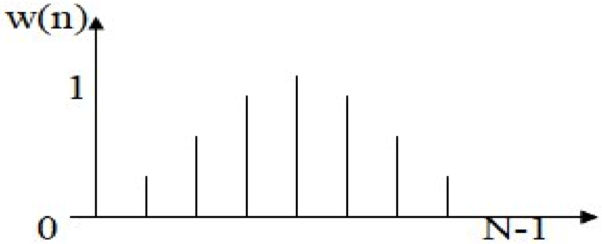
{ 0 $\quad$ elsewhere



3-Hanning:

$$w(n) = 0.5 - 0.5\cos(\frac{2\pi n}{N-1}) \quad N-1 \geq n \geq 0$$

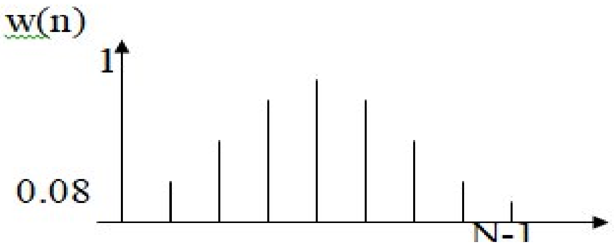
0 elsewhere



4-Hamming:

$$w(n) = 0.54 - 0.46\cos(\frac{2\pi n}{N-1}) \quad N-1 \geq n \geq 0$$

0 elsewhere

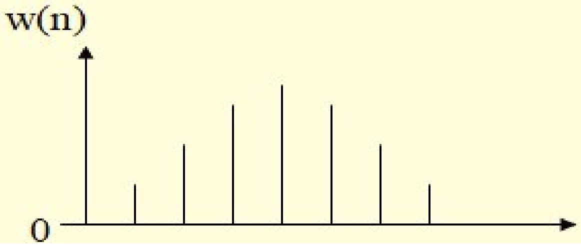


5-Blackman:

$$w(n) = 0.42 - 0.5\cos(\frac{2\pi n}{N-1}) + 0.08\cos(\frac{4\pi n}{N-1})$$

For N-1≥ n≥ 0

0 elsewhere



Now to find the overall impulse response h(n) with windowing, then:

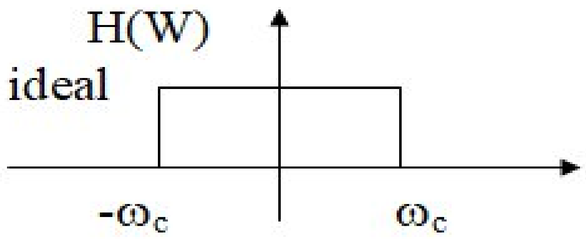
$$\mathbf{h(n) = h_d(n)\, w(n)}$$

where ***$h_d(n)$*** depends on filter type (LFP,HPF,BPF,BSF..) and w(n) is usually chosen according to the required sidelobe level (SLL) at stopband . To find ***$h_d(n)$,*** then:

1- For LPF with cutoff at $\omega = \omega_c$:

Since even symmetry in ω, then:

$$h_d(n) = \frac{1}{\pi}\int_0^{w_c} \cos(\omega(n-\alpha))\,d\omega$$

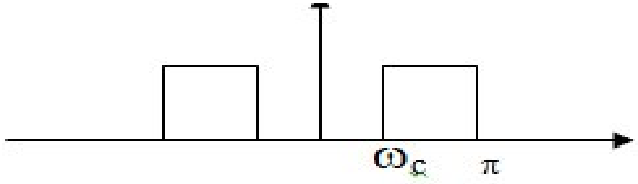


$$h_d(n) = \frac{\sin\{\omega_c(n-\alpha)\}}{\pi(n-\alpha)} \quad \text{For } N-1 \ge n \ge 0,\ n \ne \alpha$$

And $h_d(\alpha) = \omega_c/\pi$

2- For HPF with cutoff at $\omega = \omega_c$:

$$h_d(n) = \frac{1}{\pi}\int_{\omega_c}^{\pi} \cos(\omega(n-\alpha))\,d\omega$$



$$h_d(n) = -\frac{\sin\{\omega_c(n-\alpha)\}}{\pi(n-\alpha)} \quad \text{For } N-1 \ge n \ge 0,\ n \ne \alpha$$

And $h_d(\alpha) = 1 - (\omega_c/\pi)$

note: the maximum used digital frequency is 🚔 Since $f_s(\min) = 2f$ according to Nyquist rate

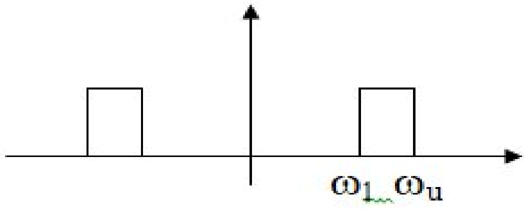
3- <u>For HPF with lower and upper cutoff frequencies ω_ℓ and ω_u</u>:

$$h_d(n) = \frac{1}{\pi}\int_{\omega_\ell}^{\omega_u} \cos(\omega(n-\alpha))d\omega$$

$$h_d(n) = \frac{\sin\{\omega_u(n-\alpha)\} - \sin\{\omega_\ell(n-\alpha)\}}{\pi(n-\alpha)}$$

for $N\text{-}1 \geq n \geq 0,\ n \neq \alpha$

And $h_d(\alpha) = (\omega_u - \omega_\ell)/\pi$



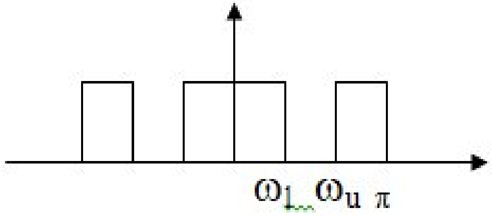
4- <u>For BSF with lower and upper cutoff frequencies ω_ℓ and ω_u</u>:

$$h_d(n) = \frac{1}{\pi}\int_{\omega_\ell}^{\omega_u} \cos(\omega(n-\alpha))d\omega$$

$$h_d(n) = \frac{\sin\{\omega_\ell(n-\alpha)\} - \sin\{\omega_u(n-\alpha)\}}{\pi(n-\alpha)}$$

for $N\text{-}1 \geq n \geq 0,\ n \neq \alpha$

And $h_d(\alpha) = (\pi - \omega_u + \omega_\ell)/\pi$



In general, for all above types of filters with N odd, then:

$\theta(\omega) = -\alpha\omega$ (linear phase) and

$$|H(\omega)| = h(\alpha) + \sum_{n=0}^{\alpha-1} 2h(n)\cos(\omega(n-\alpha))$$

<u>Next, we discuss how to find N</u>: usually, N gives filter roll of (how fast is the transition from passband to stopband or how is the obtained response is closed to ideal response). In general

$$N = \frac{2\pi\ (K\ factor)}{(\omega_2 - \omega_1)} \qquad \text{greatest odd integer}$$

where k_1 and k_2 are gain values at ω_1 and ω_2 and k_1 is the -3dB level and k_2 is usually taken as the maximum SLL.

<u>Example</u> : Design and realize a linear phase digital lowpass filter (LPF) having 3dB cutoff frequency of 7.5KHz. and stopband attenuation of at least 40dB at 35KHz. Find the difference equation and the frequency response of this filter. Use f_s = 100KHz

<u>Solution:</u>

This is FIR filter since linear phase is required. Since k_2= -40dB, this gives maxi SLL from which type of window is chosen according to to previous window-type table where Hanning window is chosen since its max. SLL is -44dB,

Next, we find N:

If f_1=7.5KHz, then $\omega_1 = 2\pi(7.5)/100 = 0.15\pi$ rad (recall that $\omega = 2\pi f/f_s$)

Similarly, if f_2=35KHz, then $\omega_2 = 2\pi(35)/100 = 0.7\pi$ rad.

Then: $N = \dfrac{2\pi\ (K\ \ factor)}{(\omega_2 - \omega_1)} = \dfrac{2\pi\ (4)}{0.7\pi - 0.15\pi} = 14.5 = 15$

(K factor=4 for Hanning window)

Next, we find the midpoint α=(N-1)/2=(15-1)/2=7. Here $\omega_c = \omega_1 = 0.15\pi$ rad (-3dB point).

For Hanning window $w(n) = 0.5 - 0.5\cos(\frac{2\pi n}{N-1}) = 0.5 - 0.5\cos(\frac{n\pi}{7})$

For LPF, then $h_d(n) = \frac{\sin\{\omega_c(n-\alpha)\}}{\pi(n-\alpha)} = \frac{\sin(0.15\pi(n-7))}{\pi(n-7)}$, $14 \geq n \geq 0$,

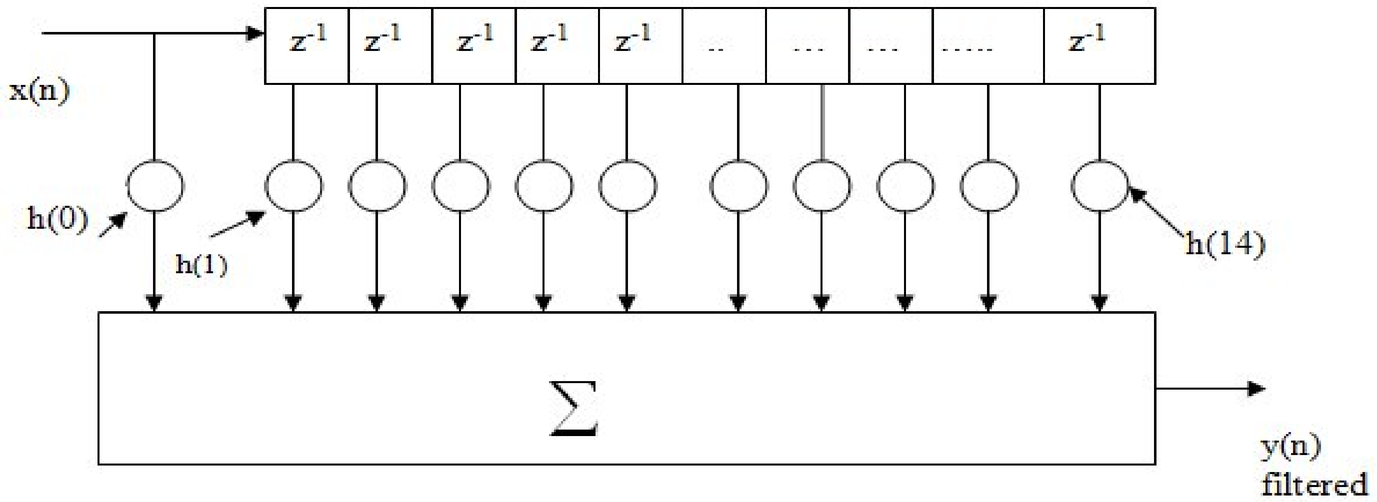
$h_d(7) = \omega_c/\pi = 0.15$. and $h(n)=h_d(n)\,w(n)$

$h(n) = \frac{\sin(0.15\pi(n-7))}{\pi(n-7)}[0.5 - 0.5\cos(\frac{n\pi}{7})]$ for $14 \geq n \geq 0,\ n \neq 7$ And $h(7)=h_d(7)= 0.15$ since $w(7)=1$.

Due to symmetry around n=7, only h(n) values at n=0,1,2,3,4,5,6 need to be calculated

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
h(n)	0	0.0008	0.0083	0.029	0.064	0.1	0.13	0.15	0.13	0.1	0.064	0.029	0.0083	0.0008	0

The realization of this filter is done using a tapped delay lines



We can also find the difference equation as:

$y(n)=x(n)\,h(0)+x(n-1)\,h(1)+x(n-2)\,h(n-2)+\ldots x(n-14)\,h(14)$

and the frequency response: $\theta(\omega) = -7\omega$ rad

$$|H(\omega)| = 0.15 + \sum_{n=0}^{6} 2h(n)\cos(\omega(n-7))$$

Example: Find the impulse response of a digital linear phase bandstop filter (BSF) having: 1- 3dB cutoff at 20Hz 1400Hz. 2- stopband attenuation of at least 40dB at 660Hz and 750Hz. Use f_s=3KHz.

Solution: since linear phase, then this is FIR filter. For 40dB SLL, we use Hanning window,

[K factor=4 and $w(n) = 0.5 - 0.5\cos(\frac{2\pi n}{N-1})$

For lowpass section:

$$N_1 = \frac{2\pi(K \quad factor)}{(\omega_2 - \omega_1)} = \frac{2\pi(4)}{\frac{2\pi(660)}{3000} - \frac{2\pi(20)}{3000}} = 18.75 \quad = 19$$

For highpass section:

$$N_2 = \frac{2\pi(K \quad factor)}{(\omega_2 - \omega_1)} = \frac{2\pi(4)}{\frac{2\pi(1400)}{3000} - \frac{2\pi(750)}{3000}} = 18.46 \quad = 19$$

When we design a filter with two section we usually use: **N=max(N_1,N_2)**=max(19,19)=19. This gives α =9.

The ω_u and ω_ℓ are the 3dB cutoff of the highpass and lowpass sections respectively,

ω_u=2π(1400)/3000=(14/15)π rad

And ω_ℓ= 2π(20)/3000=π/75 rad

$$h_d(n) = \frac{\sin\{(\pi/75)(n-9)\} - \sin\{(14/15)\pi(n-9)\}}{\pi(n-9)}$$ for 18 $\ge$ n $\ge$ 0, n≠9

And $h_d(9)$= [π-(14/15)π+(π/75)]/π=0.08

$$h(n) = [0.5 - 0.5\cos(n\pi/9)][\frac{\sin\{(\pi/75)(n-9)\} - \sin\{(14/15)\pi(n-9)\}}{\pi(n-9)}]$$

h(9)=0.08 since w(9)=1.

n	0	1	2	3	4	5	6	7	8	9	10
h(n)	0	0.0015	-0.003	0.0159	-0.0173	0.0424	-0.0368	0.0689	-0.0512	0.08	-0.0512

11	12	13	14	15	16	17	18
0.0689	-0.0368	0.0424	-0.0173	0.0159	-0.003	0.0015	0