Tikrit university

Collage of Engineering Shirqat

Department of Electrical Engineering

Fourth Class- Semester-2

Control Engineering

Chapter 8

Lecture 1

Root Locus Techniques

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8.1 Introduction

Root locus, a graphical presentation of the closed-loop poles as a system parameter is varied, is a powerful method of analysis and design for stability and transient response.

The root locus covered in this chapter is a graphical technique that gives us the qualitative description of a control system's performance that we are looking for and also serves as a powerful quantitative tool that yields more information than the methods already discussed.

Up to this point, gains and other system parameters were designed to yield a desired transient response for only first- and second-order systems. Even though the root locus can be used to solve the same kind of problem, its real power lies in its ability to provide solutions for systems of order higher than 2. For example, under the right conditions, a fourth-order system's parameters can be designed to yield a given percent overshoot and settling time using the concepts learned in previous Chapters.

Before presenting root locus, let us review two concepts that we need for the ensuing discussion: (1) the control system problem and (2) complex numbers and their representation as vectors.

8.1.1 The Control System Problem

The poles of the open-loop transfer function are easily found (typically, they are known by inspection and do not change with changes in system gain), the poles of the closed-loop transfer function are more difficult to find (typically, they cannot be found without factoring the closed-loop system's characteristic polynomial, the denominator of the closed-loop transfer function), and further, the closed-loop poles change with changes in system gain.

A typical closed-loop feedback control system is shown in Figure 8.1(a). The open-loop transfer function is $KG(s)H(s)$. Ordinarily, we can determine the poles of $KG(s)H(s)$, since these poles arise from simple cascaded first- or second-order subsystems. Further, variations in K do not affect the location of any pole of this function. On the other hand, we cannot determine the poles of T(s) = KG(s)/[1 + KG(s)H(s)] unless we factor the denominator. Also, the poles of T(s) change with K.

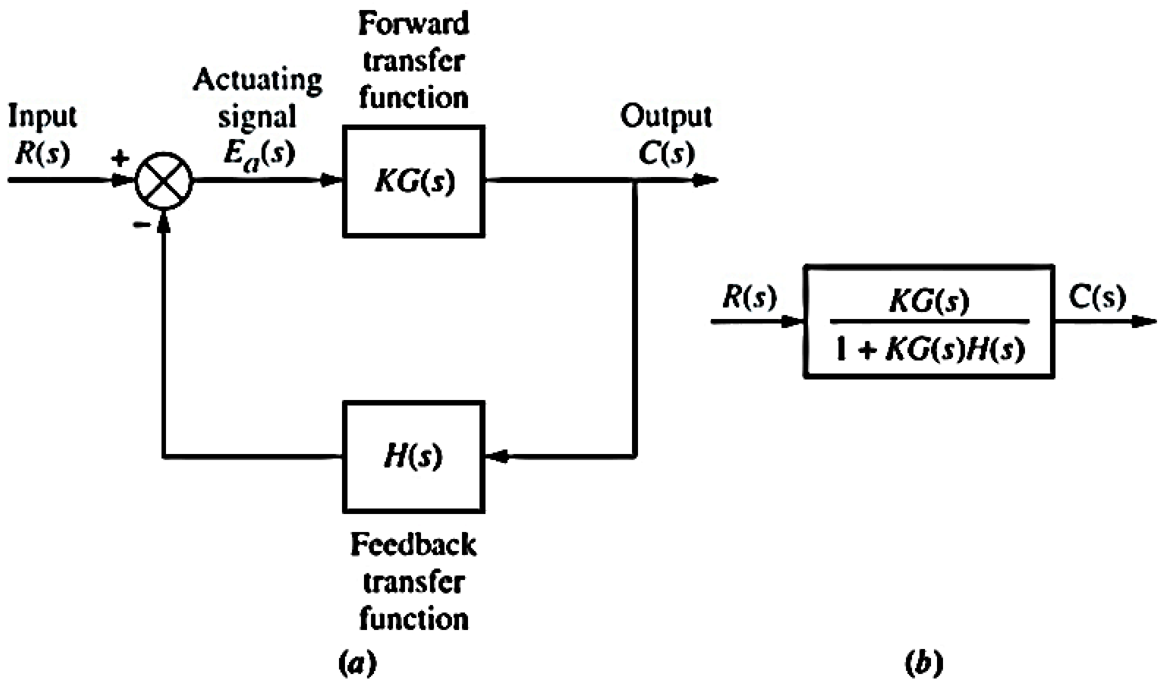


FIGURE 8.1 a. Closed-loop system; b. equivalent transfer function

Let us demonstrate. Letting

$$G(s) = \frac{N_G(s)}{D_G(s)} \tag{8.1}$$

and

$$H(s) = \frac{N_H(s)}{D_H(s)} \tag{8.2}$$

then

$$\boxed{T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s)+KN_G(s)N_H(s)}} \tag{8.3}$$

where N and D are factored polynomials and signify numerator and denominator terms, respectively. We observe the following: Typically, we know the factors of the numerators and denominators of $G(s)$ and $H(s)$. Also, the zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$. The poles of $T(s)$ are not immediately known and in fact can change with K. For example, if $G(s) = (s + 1)/[s\,(s + 2)]$ and $H(s) = (s + 3)/(s + 4)$, the poles of $KG(s)H(s)$ are 0, − 2, and − 4. The zeros of $KG(s)H(s)$ are −1 and − 3. Now, $T(s) = K(s + 1)(s + 4)/[s^3 + (6 + K)s^2 + (8 + 4K)s + 3K]$. Thus, the zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$. The poles of $T(s)$ are not immediately known without factoring the denominator, and they are a function of K. Since the system's transient response and stability are dependent upon the poles of $T(s)$, we have no knowledge of the system's performance unless we factor the denominator for specific values of K. The root locus will be used to give us a vivid picture of the poles of $T(s)$ as K varies.

8.1.2 Vector Representation of Complex Numbers

Any complex number, $\sigma + j\omega$, described in Cartesian coordinates can be graphically represented by a vector, as shown in Figure 8.2(a). The complex number also can be described in polar form with magnitude M and angle θ, as $M\angle\theta$. If the complex number is substituted into a complex function, F(s), another complex number will result. For example, if $F(s) = (s + a)$, then substituting the complex number $s = \sigma + j\omega$ yields $F(s) = (\sigma + a) + j\omega$, another complex number. This number is shown in Figure 8.2(b). Notice that F(s) has a zero at −a. If we translate the vector a unit to the left, as in Figure 8.2(c), we have an alternate representation of the complex number that originates at the zero of F(s) and terminates on the point $s = \sigma + j\omega$.

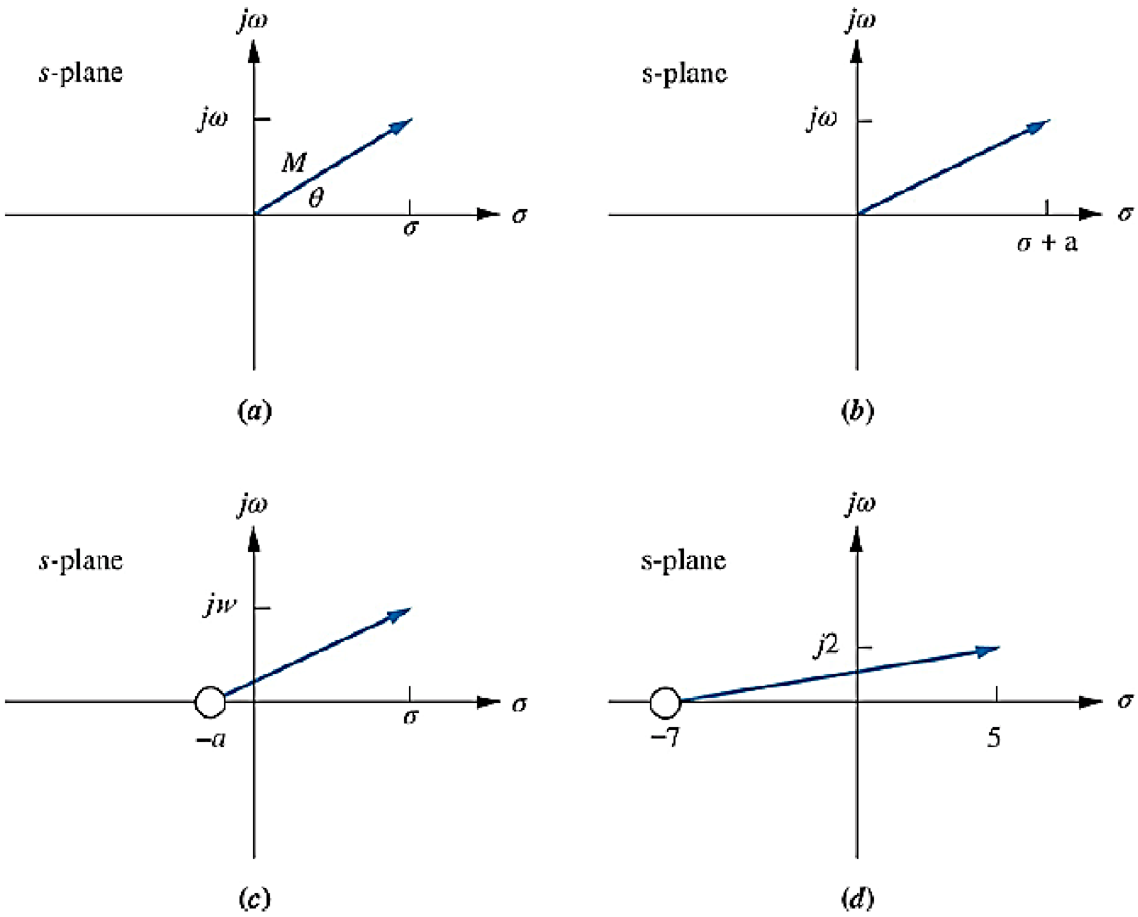


FIGURE 8.2 Vector representation of complex numbers: a. s= σ + jω; b.(s + a); c. alternate representation of (s + a); d.(s + 7) $|_{s\to 5+j2}$

We conclude that (s + a) is a complex number and can be represented by a vector drawn from the zero of the function to the point s. For example, (s + 7) $|_{s\to 5+j2}$ is a complex number drawn from the zero of the function, -7, to the point s, which is 5 + j2, as shown in Figure 8.2(d).

Now let us apply the concepts to a complicated function. Assume a function

$$F(s)=\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=\frac{\Pi\,\text{numerator's complex factors}}{\Pi\,\text{denominator's complex factors}} \tag{8.4}$$

where the symbol Π means "product," m = number of zeros, and n = number of poles. Each factor in the numerator and each factor in the denominator is a complex number that can be represented as a vector. The function defines the complex arithmetic to be performed in order to evaluate $F(s)$ at any point, s. Since each complex factor can be thought of as a vector, the magnitude, M, of $F(s)$ at any point, s, is

$$\boxed{M = \frac{\Pi \text{ zero lengths}}{\Pi \text{ pole lengths}} = \frac{\prod_{i=1}^{m} |(s+z_i)|}{\prod_{j=1}^{n} |(s+p_j)|}} \quad (8.5)$$

where a zero length, $|(s + z_i)|$, is the magnitude of the vector drawn from the zero of $F(s)$ at $-z_i$ to the point s, and a pole length, $|(s + p_j)|$, is the magnitude of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point s. The angle, θ, of $F(s)$ at any point, s, is

$$\boxed{\begin{aligned}\theta &= \sum \text{zero angles} - \sum \text{pole angles} \\ &= \sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_j)\end{aligned}} \quad (8.6)$$

where a zero angle is the angle, measured from the positive extension of the real axis, of a vector drawn from the zero of $F(s)$ at $-z_i$ to the point s, and a pole angle is the angle, measured from the positive extension of the real axis, of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point s.As a demonstration of Eqs. (8.5) and (8.6), consider the following example.

Example 8.1 Evaluation of a Complex Function via Vectors

PROBLEM:

Given

$$F(s) = \frac{(s+1)}{s(s+2)} \tag{8.7}$$

find $F(s)$ at the point $s = -3 + j4$.

SOLUTION:

The problem is graphically depicted in Figure 8.3, where each vector, $(s + \alpha)$, of the function is shown terminating on the selected point $s = -3 + j4$. The vector originating at the zero at -1 is

$$\sqrt{20}\angle 116.6^{\circ} \tag{8.8}$$

The vector originating at the pole at the origin is

$$5\angle 126.9^{\circ} \tag{8.9}$$

The vector originating at the pole at -2 is

$$\sqrt{17}\angle 104.0^{\circ} \tag{8.10}$$

Substituting Eqs. (8.8) through (8.10) into Eqs. (8.5) and (8.6) yields

$$M\angle\theta = \frac{\sqrt{20}}{5\sqrt{17}}\angle 116.6^{\circ} - 126.9^{\circ} - 104.0^{\circ} = 0.217\angle - 114.3^{\circ} \tag{8.11}$$

as the result for evaluating $F(s)$ at the point $-3 + j4$.

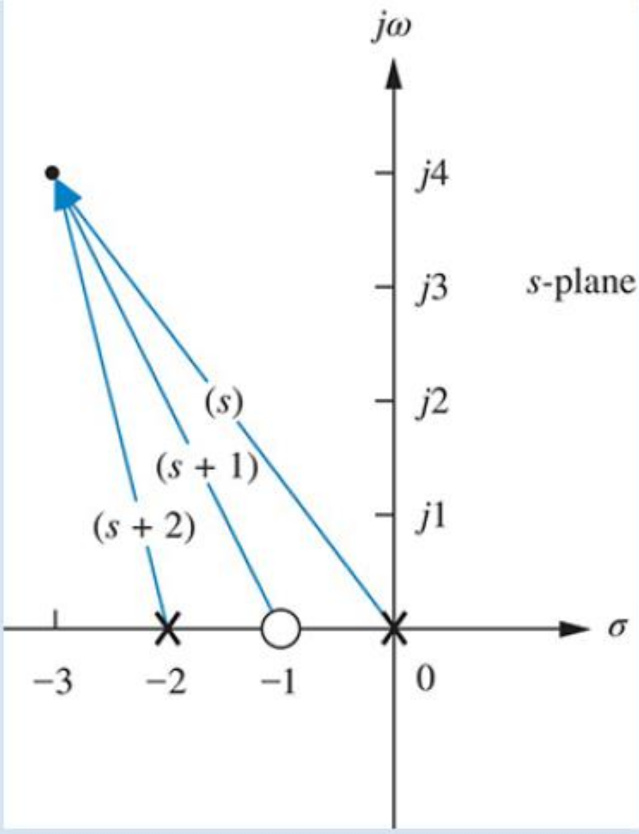
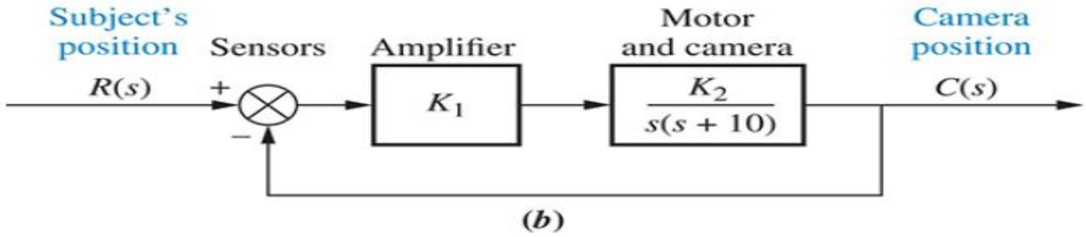


FIGURE 8.3 Vector representation of Eq. (8.7)

8.2 Defining the Root Locus

A security camera system similar to that shown in Figure 8.4(a) can automatically follow a subject. The tracking system monitors pixel changes and positions the camera to center the changes.

(*a*)



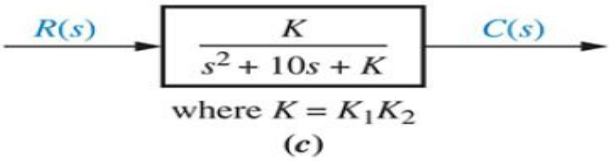


FIGURE 8.4 a. Security cameras with auto tracking can be used to follow moving objects automatically; b. block diagram; c. closed-loop transfer function

The root locus technique can be used to analyze and design the effect of loop gain upon the system's transient response and stability. Assume the block diagram representation of a tracking system as shown in Figure 8.4(b), where the closed-loop poles of the system change location as the gain, K, is varied. Table 8.1, which was formed by applying the quadratic formula to the denominator of the transfer function in Figure 8.4(c), shows the variation of pole location for different values of gain, K. The data of Table 8.1 is graphically displayed in Figure 8.5(a), which shows each pole and its gain.

TABLE 8.1

Pole location as function of gain for the system of Figure 8.4

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5+j2.24$	$-5-j2.24$
35	$-5+j3.16$	$-5-j3.16$
40	$-5+j3.87$	$-5-j3.87$
45	$-5+j4.47$	$-5-j4.47$
50	$-5+j5$	$-5-j5$

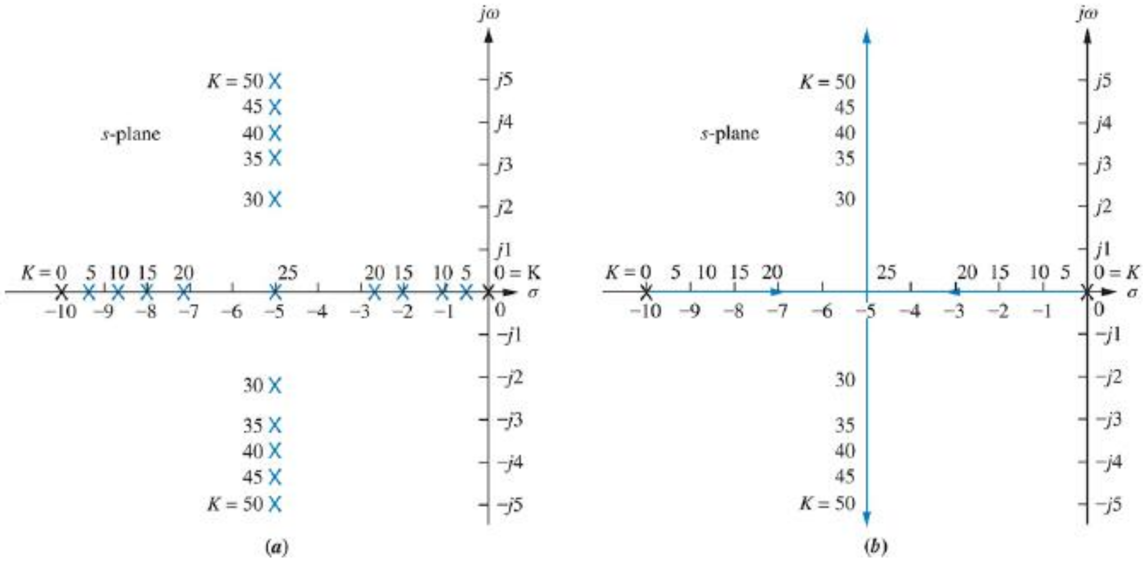


FIGURE 8.5 a. Pole plot from Table 8.1; b. root locus

As the gain, *K*, increases in Table 8.1 and Figure 8.5(*a*), the closed-loop pole, which is at −10 for *K* = 0, moves toward the right, and the closed-loop pole, which is at 0 for *K* = 0, moves toward the left. They meet at −5, break away from the real axis, and move into the complex plane. One closed-loop pole moves upward while the other moves downward. We cannot tell which pole moves up or which moves down. In Figure 8.5(*b*), the individual closed-loop pole locations are removed and their paths are represented with solid lines. It is this *representation of the paths of the closed-loop poles as the gain is varied* that we call a ***root locus***. For most of our work the discussion will be limited to positive gain, or $K \geq 0$

8.3 Properties of the Root Locus

In Section 8.2, we arrived at the root locus by factoring the second order polynomial in the denominator of the transfer function. Consider what would happen if that polynomial were of fifth or tenth order. Without a computer, factoring the polynomial would be quite a problem for numerous values of gain. We are about to examine the properties of the root locus. From these properties we will be able to make a rapid *sketch* of the root locus for higher-order systems without having to factor the denominator of the closed-loop transfer function. The properties of the root locus can be derived from the general control system of Figure 8.1(*a*). The closed-loop transfer function for the system is

$$\boxed{T(s) = \frac{KG(s)}{1+KG(s)H(s)}} \quad (8.12)$$

From Eq. (8.12), a pole, *s*, exists when the characteristic polynomial in the denominator becomes zero, or

$$\boxed{KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ \quad k = 0, \pm1, \pm2, \pm3, \ldots} \quad (8.13)$$

where −1 is represented in polar form as $1\angle(2k + 1)180°$. Alternately, a value of *s* is a closed-loop pole if

$$\boxed{|KG(s)H(s)| = 1} \quad (8.14)$$

and

$$\boxed{\angle KG(s)H(s) = (2k+1)180^\circ} \quad (8.15)$$

Equation (8.13) implies that if a value of *s* is substituted into the function *KG*(*s*)*H*(*s*), a complex number result. If the angle of the complex number is an odd multiple of 180°, that value of *s* is a system pole for some particular value of *K*. What value of *K*? Since the angle criterion of Eq. (8.15) is satisfied, all that remains is to satisfy the magnitude criterion, Eq. (8.14). Thus,

$$\boxed{K = \frac{1}{|G(s)||H(s)|}} \qquad (8.16)$$

We have just found that a pole of the closed-loop system causes the angle of KG(s)H(s), or simply G(s)H(s) since K is a scalar, to be an odd multiple of 180°. Furthermore, the magnitude of KG(s)H(s) must be unity, implying that the value of K is the reciprocal of the magnitude of G(s)H(s) when the pole value is substituted for s. Let us demonstrate this relationship for the second-order system of Figure 8.4. The fact that closed-loop poles exist at −9.47 and −0.53 when the gain is 5 has already been established in Table 8.1. For this system,

$$KG(s)H(s) = \frac{K}{s(s+10)} \qquad (8.17)$$

Substituting the pole at −9.47 for s and 5 for K yields KG(s)H(s) = − 1. The student can repeat the exercise for other points in Table 8.1 and show that each case yields KG(s)H(s) = − 1.

It is helpful to visualize graphically the meaning of Eq. (8.15). Let us apply the complex number concepts reviewed in Section 8.1 to the root locus of the system shown in Figure 8.6. For this system the open-loop transfer function is

$$KG(s)H(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)} \qquad (8.18)$$

The closed-loop transfer function, T(s), is

$$T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)} \tag{8.19}$$

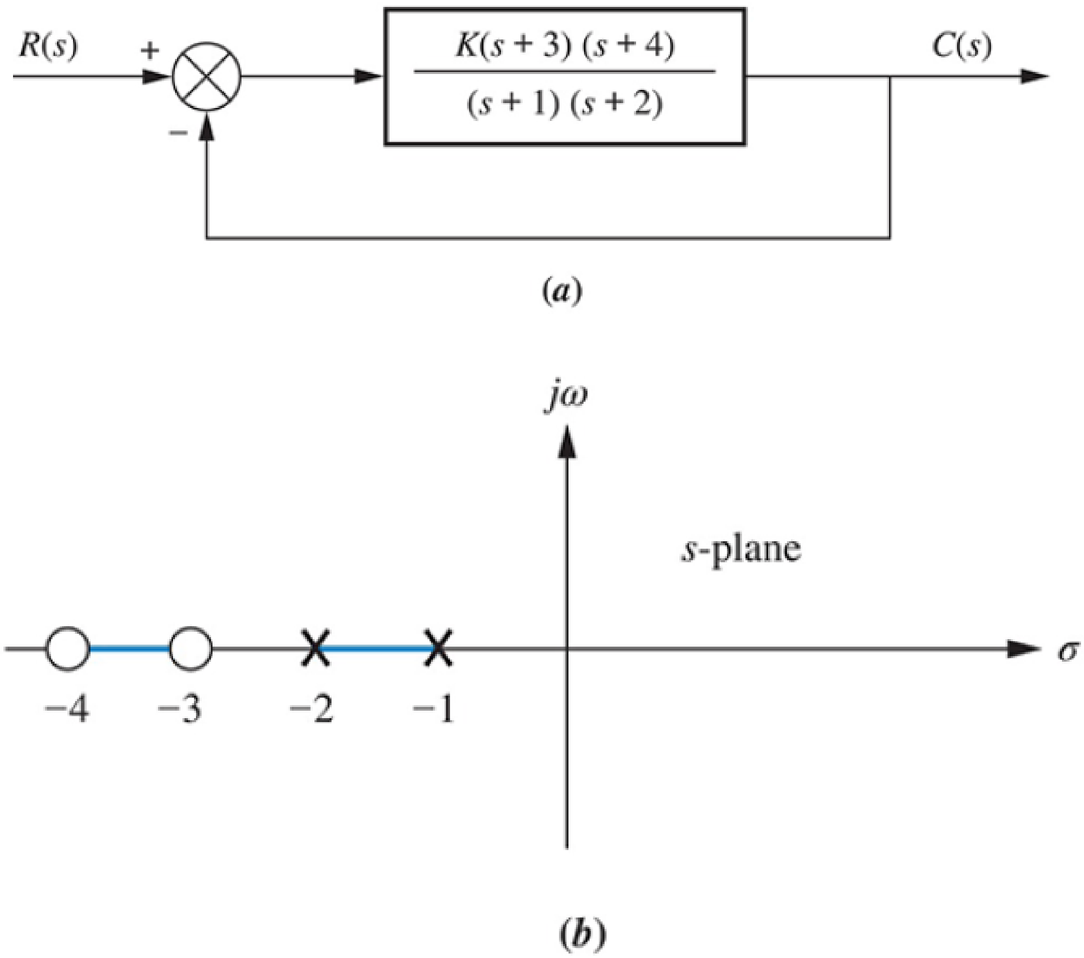


FIGURE 8.6 a. Example system; b. pole-zero plot of *G*(*s*)

If point s is a closed-loop system pole for some value of gain, K, then s must satisfy Eqs. (8.14) and (8.15). Consider the point −2 + j3. If this point is a closed-loop pole for some value of gain, then the angles of the zeros minus the angles of the poles must equal an odd multiple of 180°. From Figure 8.7,

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ = -70.55^\circ \tag{8.20}$$

Therefore, −2 + *j*3 is not a point on the root locus, or alternatively, −2 + *j*3 is not a closed-loop pole for any gain.

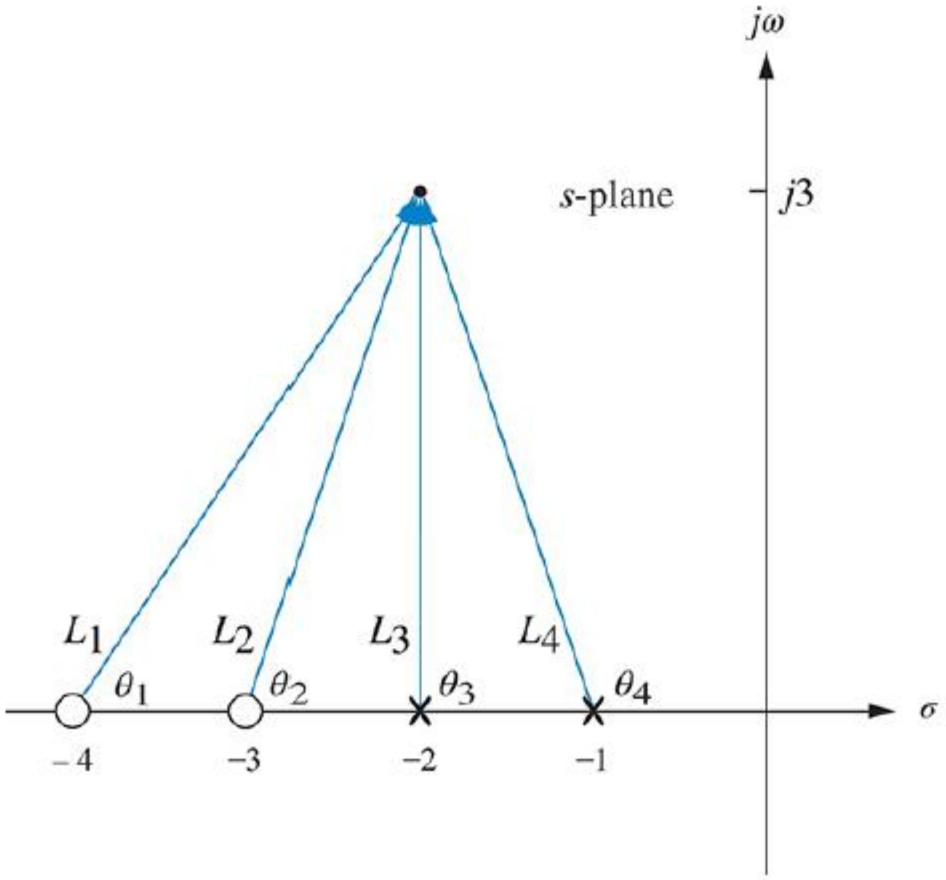


FIGURE 8.7 Vector representation of G(s) from Figure 8.6(a) at −2 + j3

If these calculations are repeated for the point −2 + j (√2/2), the angles do add up to 180°. That is, −2 + j (√2/2) is a point on the root locus for some value of gain. We now proceed to evaluate that value of gain. From Eqs. (8.5) and (8.16),

$$\boxed{K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\Pi \text{ pole lengths}}{\Pi \text{ zero lengths}}} \quad (8.21)$$

Looking at Figure 8.7 with the point −2 + *j*3 replaced by −2 + j (√2/2), the gain, *K*, is calculated as

$$K = \frac{L_3 L_4}{L_1 L_2} = \frac{\frac{\sqrt{2}}{2}(1.22)}{(2.12)\,(1.22)} = 0.33 \quad (8.22)$$

Thus, the point −2 + j (√2/2) is a point on the root locus for a gain of 0.33. We summarize what we have found as follows: Given the poles and zeros of the open-loop transfer function, KG(s)H(s), a point in the s plane is on the root locus for a particular value of gain, K, if the angles of the zeros minus the angles of the poles, all drawn to the selected point on the s-plane, add up to (2k + 1)180°. Furthermore, gain K at that point for which the angles add up to (2k + 1)180° is found by dividing the product of the pole lengths by the product of the zero lengths.

8.4 Sketching the Root Locus

It appears from our previous discussion that the root locus can be obtained by sweeping through every point in the s-plane to locate those points for which the angles, as previously described, add up to an odd multiple of 180°. Although this task is tedious without the aid of a computer, the concept can be used to develop rules that can be used to sketch the root locus without the effort required to plot the locus. Once a sketch is obtained, it is possible to accurately plot just those points that are of interest to us for a particular problem.

The following five rules allow us to sketch the root locus using minimal calculations. The rules yield a sketch that gives intuitive insight into the behaviour of a control system. In the next section, we refine the sketch by finding actual points or angles on the root locus

1. **Number of branches.** *The number of branches of the root locus equals the number of closed-loop poles.* As an example, look at Figure 8.5(*b*), where the two branches are shown. One originates at the origin, the other at −10.

2. Symmetry. *The root locus is symmetrical about the real axis.* An example of symmetry about the real axis is shown in Figure 8.5(*b*).

3. Real-axis segments. *On the real axis, for K > 0 the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros. Examine Figure 8.6(b). According to the rule just developed, the real-axis segments of the root locus are between −1 and −2 and between −3 and −4 as shown in Figure 8.9.*

4. Starting and ending points. Where does the root locus begin (zero gain) and end (infinite gain)? The answer to this question will enable us to expand the sketch of the root locus beyond the real axis segments.

The root locus begins at the finite and infinite poles of G(s)H(s) and ends at the finite and infinite zeros of G(s)H(s). Remember that these poles and zeros are the open-loop poles and zeros.

In order to demonstrate this rule, look at the system in Figure 8.6(a), whose real-axis segments have been sketched in Figure 8.9. Using the rule just derived, we find that the root locus begins at the poles at −1 and −2 and ends at the zeros at −3 and −4 (see Figure 8.10). Thus, the poles start out at −1 and −2 and move through the real-axis space between the two poles. They meet somewhere between the two poles and break out into the complex plane, moving as complex conjugates. The poles return to the real axis somewhere between the zeros at −3 and −4, where their path is completed as they move away from each other, and end up, respectively, at the two zeros of the open-loop system at −3 and −4.

5. **Behaviour at infinity**. Consider applying Rule 4 to the following open-loop transfer function:

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)} \qquad (8.25)$$

There are three finite poles, at s = 0, − 1, and − 2, and no finite zeros.

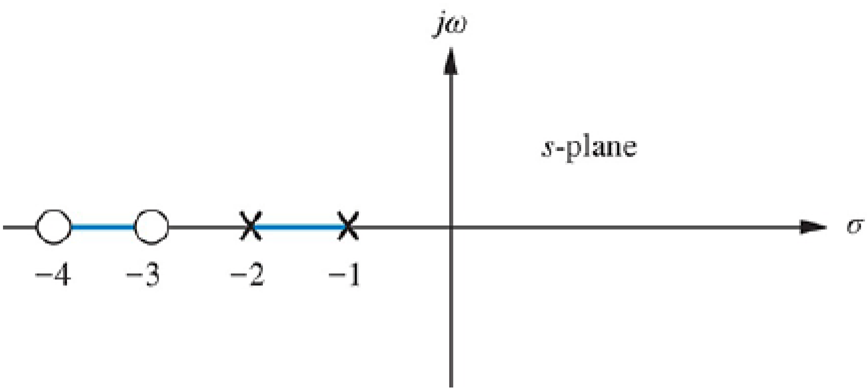


FIGURE 8.9 Real-axis segments of the root locus for the system of Figure 8.6

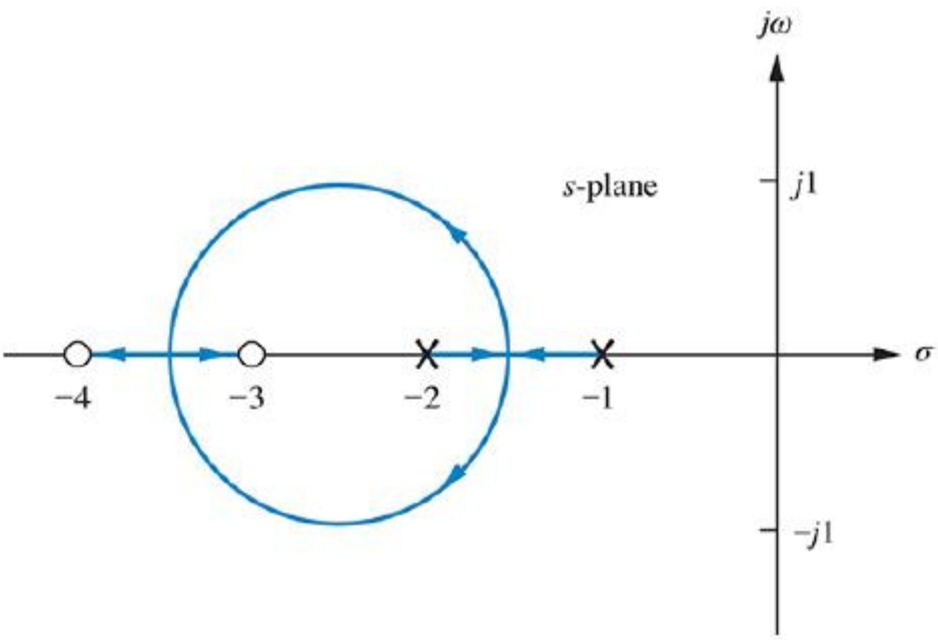


FIGURE 8.10 Complete root locus for the system of Figure 8.6

Every function of *s* has an equal number of poles and zeros if we include the infinite poles and zeros as well as the finite poles and zeros. In this example, Eq. (8.25) contains three finite poles and three infinite zeros. To illustrate, let *s* approach infinity. The open-loop transfer function becomes

$$KG(s)H(s) \approx \frac{K}{s^3} = \frac{K}{s \cdot s \cdot s} \qquad (8.26)$$

Each *s* in the denominator causes the open-loop function, $KG(s)H(s)$, to become zero as that *s* approaches infinity. Hence, Eq. (8.26) has three zeros at infinity.

We now state Rule 5, which will tell us what the root locus looks like as it approaches the zeros at infinity or as it moves from the poles at infinity.

The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a and angle, θ_a as follows:

$$\boxed{\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}} \quad (8.27)$$

$$\boxed{\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}} \quad (8.28)$$

where k = 0, ±1, ±2, ±3 and the angle is given in radians with respect to the positive extension of the real axis. Notice that the running index, k, in Eq. (8.28) yields a multiplicity of lines that account for the many branches of a root locus that approach infinity. Let us demonstrate the concepts with an example

Example 8.2 Sketching a Root Locus with Asymptotes

PROBLEM:

Sketch the root locus for the system shown in Figure 8.11.

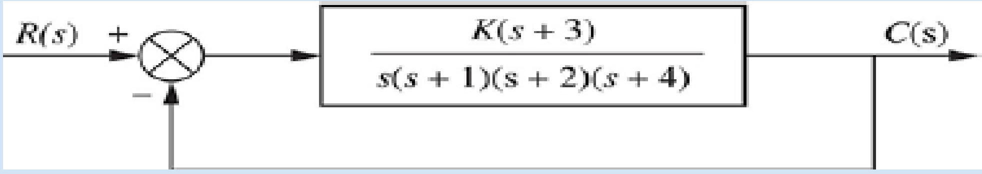


FIGURE 8.11 System for Example 8.2

SOLUTION:

Let us begin by calculating the asymptotes. Using Eq. (8.27), the real-axis intercept is evaluated as

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3} \quad (8.29)$$

The angles of the lines that intersect at −4/3, given by Eq. (8.28), are

$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \quad (8.30a)$$

$$= \pi/3 \quad \text{for } k = 0 \quad (8.30b)$$

$$= \pi \quad \text{for } k = 1 \quad (8.30c)$$

$$= 5\pi/3 \quad \text{for } k = 2 \quad (8.30d)$$

If the value for k continued to increase, the angles would begin to repeat. The number of lines obtained equals the difference between the number of finite poles and the number of finite zeros.

Rule 4 states that the locus begins at the open-loop poles and ends at the open-loop zeros. For the example there are more open-loop poles than open-loop zeros. Thus, there must be zeros at infinity. The asymptotes tell us how we get to these zeros at infinity. Figure 8.12 shows the complete root locus as well as the asymptotes that were just calculated. Notice that we have made use of all the rules learned so far. The real-axis segments lie to the left of an odd number of poles and/or zeros. The locus starts at the open-loop poles and ends at the open-loop zeros. For the example there is only one open-loop finite zero and three infinite zeros. Rule 5, then, tells us that the three zeros at infinity are at the ends of the asymptotes.

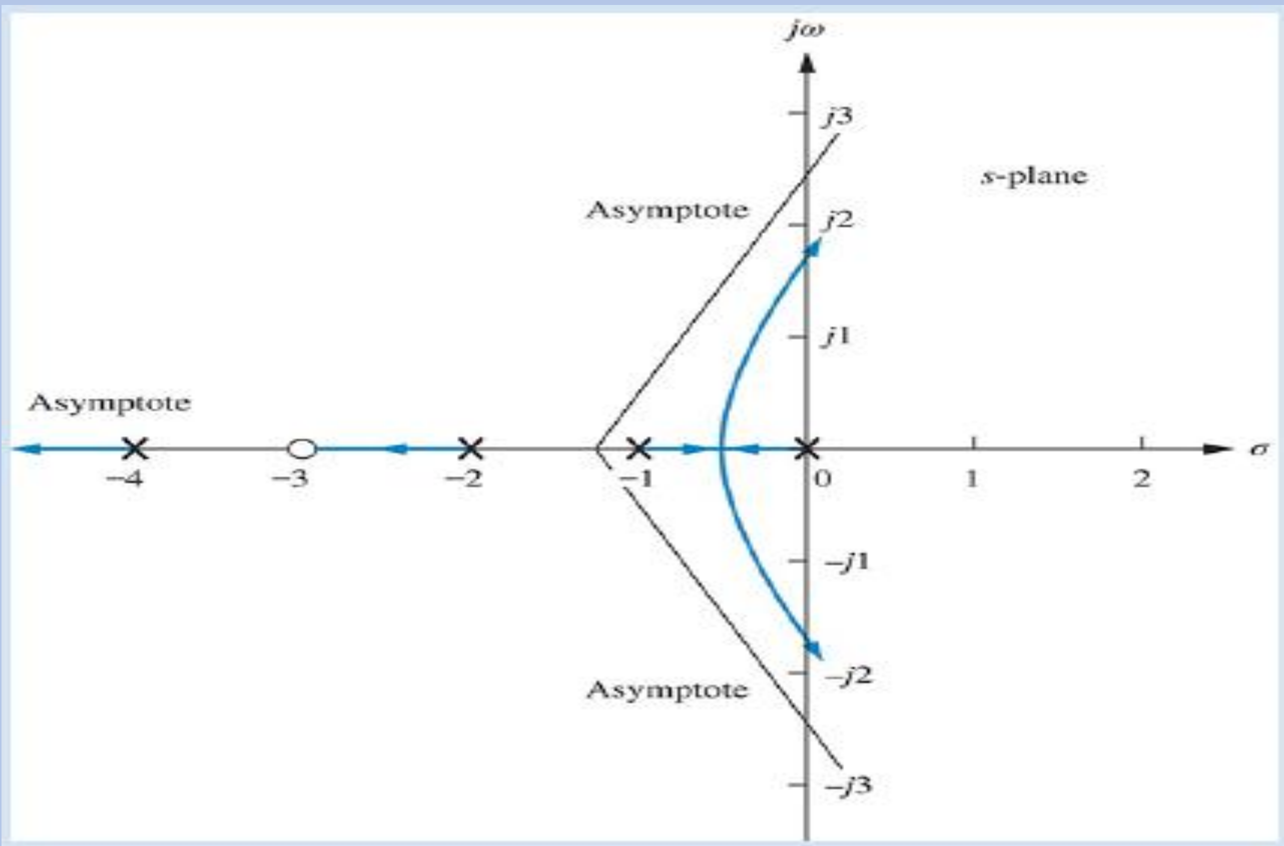


FIGURE 8.12 Root locus and asymptotes for the system of Figure 8.11