Tikrit university

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Department of Electrical Engineering

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Control Engineering

Chapter 2

Transfer functions

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2.2Electrical Network Transfer Functions

In this section, we formally apply the transfer function to the mathematical modelling of electric circuits including passive networks and operational amplifier circuits. Subsequent sections cover mechanical and electromechanical systems. Equivalent circuits for the electric networks that we work with first consist of three passive linear components: resistors, capacitors, and inductors. Table 2.3 summarizes the components and the relationships between voltage and current and between voltage and charge under zero initial conditions.

TABLE 2.3

Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage–current	Current–voltage	Voltage–charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)\,d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: v(t) − V (volts), i(t) − A (amps), q(t) – Q (coulombs), C − F (farads), R − Ω (ohms), G − Ω (mhos), L − H (Henries).

We now combine electrical components into circuits, decide on the input and output, and find the transfer function. Our guiding principles are Kirchhoff's laws. We sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the result to zero. From these relationships we can write the differential equations for the circuit. Then we can take the Laplace transforms of the differential equations and finally solve for the transfer function.

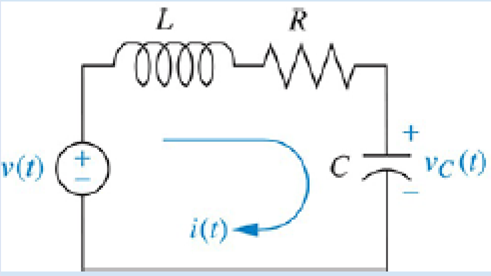
Simple Circuits via Mesh Analysis

Transfer functions can be obtained using Kirchhoff's voltage law and summing voltages around loops or meshes. We call this method *loop* or *mesh analysis* and demonstrate it in the following example.

Example 2.6 Transfer Function—Single Loop via the Differential Equation

PROBLEM:

Find the transfer function relating the capacitor voltage, *VC*(*s*), to the input voltage, *V*(*s*) in figure below



***RLC* network**

SOLUTION:

In any problem, the designer must first decide what the input and output should be. In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current. The problem statement, however, is clear in this case: We are to treat the capacitor voltage as the output and the applied voltage as the input. Summing the voltages around the loop, assuming zero initial conditions, yields the integrodifferential equation for this network as

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t) \qquad \textbf{2.12}$$

Changing variables from current to charge using i(t) = dq(t)/dt yields

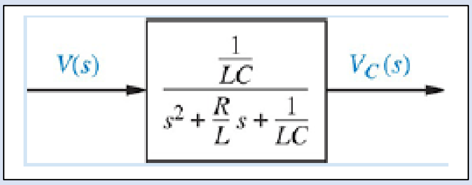
$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \qquad \textbf{2.13}$$

From the voltage–charge relationship for a capacitor in Table 2.3, $q(t) = Cv_C(t)$

By substituting yields $LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t)$ **2.14**

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields $(LCs^2 + RCs + 1)\,V_C(s) = V(s)$ **2.15**

Solving for the transfer function, $V_C(s)/V(s)$, we obtain $\dfrac{V_C(s)}{V(s)} = \dfrac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$ **2.16**



Block diagram of series *RLC* electrical network

Let us now develop a technique for simplifying the solution for future problems. First, take the Laplace transform of the equations in the voltage-current column of Table 2.3 assuming zero initial conditions. For the capacitor, $\boxed{V(s) = \dfrac{1}{Cs}I(s)}$ **2.17**

For the resistor, $\boxed{V(s) = RI(s)}$ **2.18**

For the inductor, $\boxed{V(s) = LsI(s)}$ **2.19**

Now define the following transfer function: $\boxed{\dfrac{V(s)}{I(s)} = Z(s)}$ **2.20**

Notice that this function is similar to the definition of resistance, that is, the ratio of voltage to current. But, unlike resistance, this function is applicable to capacitors and inductors and carries information on the dynamic behavior of the component, since it represents an equivalent differential equation. We call this particular transfer function *impedance*. The impedance for each of the electrical elements is shown in Table 2.3.

Let us now demonstrate how the concept of impedance simplifies the solution for the transfer function. The Laplace transform of equation $L\dfrac{di(t)}{dt} + Ri(t) + \dfrac{1}{C}\displaystyle\int_0^t i(\tau)\,d\tau = v(t)$

assuming zero initial conditions, is $\left(Ls + R + \dfrac{1}{Cs}\right)I(s) = V(s)$ **2.21**

Notice that Eq. above which is in the form

[Sum of impedances] I (s) = [Sum of applied voltages] **2.22** suggests the series circuit shown in Figure below (Laplace-transformed network). Also notice that the circuit could have been obtained immediately from the circuit of Figure above (*RLC* network) simply by replacing each element with its impedance. We call this altered circuit the *transformed circuit*. Finally, notice that the transformed circuit leads immediately to Eq. (above) if we add impedances in series as we add resistors in series. Thus, rather than writing the differential equation first and then taking the Laplace transform, we can draw the transformed circuit and obtain the Laplace transform of the differential equation simply by applying Kirchhoff's voltage law to the transformed circuit. We summarize the steps as follows:

1. Redraw the original network showing all time variables, such as $v(t)$, $i(t)$, and $vC(t)$, as Laplace transforms $V(s)$, $I(s)$, and $VC(s)$, respectively.

2. Replace the component values with their impedance values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values.

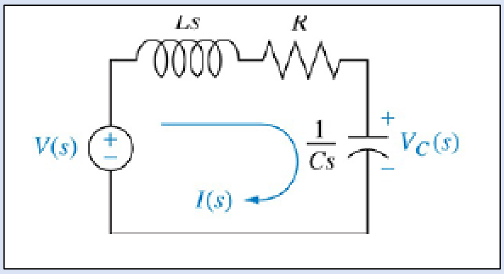
We now redo Example 2.6 using the transform methods just described and bypass the writing of the differential equation.

Example 2.7 Transfer Function—Single Loop via Transform Methods

PROBLEM: Repeat Example 2.6 using mesh analysis and transform methods without writing a differential equation.

SOLUTION:

Using Figure below and writing a mesh equation using the impedances as we would use resistor values in a purely resistive circuit, we obtain



$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s) \quad (2.23)$$

Solving for $I(s)/V(s)$,

$$\boxed{\frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}} \quad (2.24)$$

Laplace-transformed network

But the voltage across the capacitor, $VC(s)$, is the product of the current and the impedance of the capacitor. Thus,

$$V_C(s) = I(s)\frac{1}{Cs} \quad \mathbf{2.25}$$

Solving Eq. (2.25) for I(s) substituting I(s) into Eq. (2.24), and simplifying yields the same result as Eq. (2.16).

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Simple Circuits via Nodal Analysis

Transfer functions also can be obtained using Kirchhoff's current law and summing currents flowing from nodes. We call this method nodal analysis. We now demonstrate this principle by redoing Example 2.6 using Kirchhoff's current law and the transform methods just described to bypass writing the differential equation

Example 2.8 Transfer Function—Single Node via Transform Methods

PROBLEM:

Repeat Example 2.6 using nodal analysis and without writing a differential equation.

SOLUTION:

The transfer function can be obtained by summing currents flowing out of the node whose voltage is $V_C(s)$ in Figure of (**Laplace-transformed network)**

. We assume that currents leaving the node are positive and currents entering the node are negative. The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor. From Eq. (2.20), each $I(s) = V(s)/Z(s)$. Hence,

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0 \quad \mathbf{2.26}$$

where $V_C(s)/(1/Cs)$ is the current flowing out of the node through the capacitor, and $[V_C(s) - V(s)]/(R + Ls)$ is the current flowing out of the node through the series resistor and inductor.

Solving Eq. (2.26) for the transfer function, $V_C(s)/V(s)$, we arrive at the same result as Eq. (2.16).

Simple Circuits via Voltage Division

Example 2.6 can be solved directly by using voltage division on the transformed network. We now demonstrate this technique.

Example 2.9 Transfer Function—Single Loop via Voltage Division

PROBLEM: Repeat Example 2.6 using voltage division and the transformed circuit.

SOLUTION:

The voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances. Thus,

$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)} V(s) \qquad \textbf{2.27}$$

Solving for the transfer function, $VC(s)/V(s)$, yields the same result as Eq. (2.16).

Review Examples 2.6 through 2.9. Which method do you think is easiest for this circuit?

The previous example involves a simple, single-loop electrical network. Many electrical networks consist of multiple loops and nodes, and for these circuits we must write and solve simultaneous differential equations in order to find the transfer function, or solve for the output.

Complex Circuits via Mesh Analysis

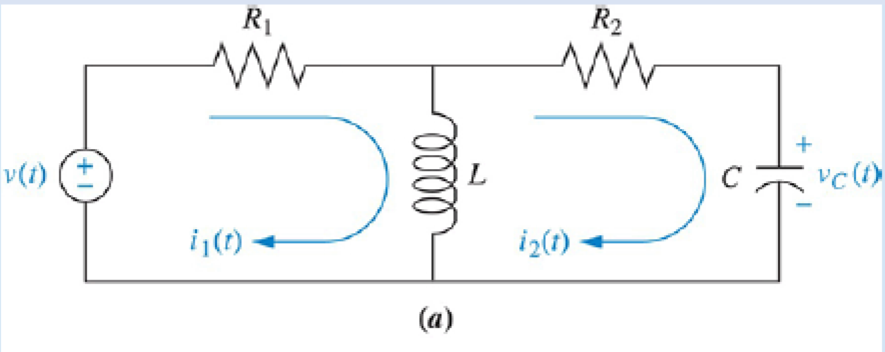
To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh.
4. Write Kirchhoff's voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Example 2.10 Transfer Function—Multiple Loops

PROBLEM:

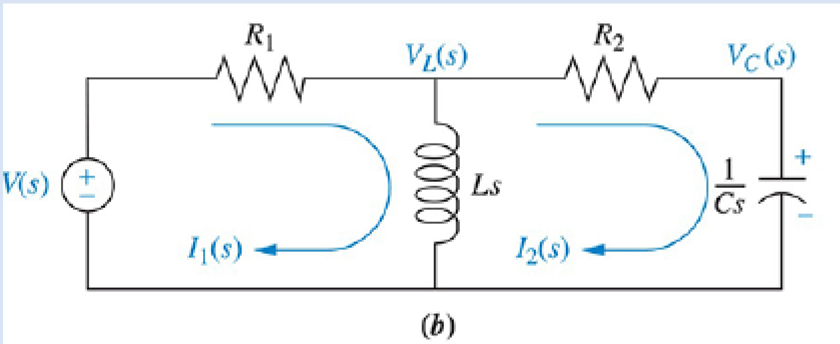
Given the network of Figure (*a*), find the transfer function, $I2(s)/V(s)$.



a. Two-loop electrical network

SOLUTION:

The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions. The result is shown in Figure (b).



b. transformed two-loop electrical network;

The circuit with which we are dealing requires two simultaneous equations to solve for the transfer function. These equations can be found by summing voltages around each mesh through which the assumed currents, $I_1(s)$ and $I_2(s)$, flow. Around Mesh 1, where $I_1(s)$ flows,

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s) \qquad \textbf{2.28}$$

Around Mesh 2, where $I2(s)$ flows,

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0 \qquad \textbf{2.29}$$

Combining terms, Eqs. (2.28) and (2.29) become simultaneous equations in $I_1(s)$ and $I_2(s)$:

$$(R_1 + Ls)I_1(s) - LsI_2(s) = V(s) \qquad \textbf{2.30a}$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) = 0 \qquad \textbf{2.30b}$$

We can use Cramer's rule (or any other method for solving simultaneous equations) to solve Eq.

(2.30) for $I_2(s)$.4 Hence,

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta} \qquad \textbf{2.31}$$

Where

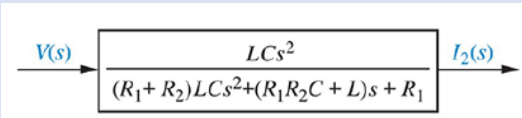
$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right) \end{vmatrix}$$

Forming the transfer function, $G(s)$, yields

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1} \qquad \textbf{2.32}$$

as shown in Figure (*c*).

We have succeeded in modelling a physical network as a transfer function: The network of Figure (*a*) is now modelled as the transfer function of Figure (*c*). Before leaving the example, we notice a pattern first illustrated by Eq. (2.22). The form that Eq. (2.30) take is



c. block diagram

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{around Mesh 1}\end{bmatrix} I_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to the}\\ \text{two meshes}\end{bmatrix} I_2(s) = \begin{bmatrix}\text{Sum of applied}\\ \text{voltages around}\\ \text{Mesh 1}\end{bmatrix} \qquad \textbf{2.33a}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to the}\\ \text{two meshes}\end{bmatrix} I_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{around Mesh 2}\end{bmatrix} I_2(s) = \begin{bmatrix}\text{Sum of applied}\\ \text{voltages around}\\ \text{Mesh 2}\end{bmatrix} \qquad \textbf{2.33b}$$

Often, the easiest way to find the transfer function is to use nodal analysis rather than mesh analysis. The number of simultaneous differential equations that must be written is equal to the number of nodes whose voltage is unknown. In the previous example we wrote simultaneous mesh equations using Kirchhoff's voltage law. For multiple nodes we use Kirchhoff's current law and sum currents flowing from each node. Again, as a convention, currents flowing from the node are assumed to be positive, and currents flowing into the node are assumed to be negative. Before progressing to an example, let us first define *admittance*, $Y(s)$, as the reciprocal of impedance, or

$$\boxed{Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}} \qquad \textbf{2.34}$$

When writing nodal equations, it can be more convenient to represent circuit elements by their admittance. Admittances for the basic electrical components are shown in Table 2.3. Let us look at an example.

Example 2.11 Transfer Function—Multiple Nodes

PROBLEM:

Find the transfer function, $V_C(s)/V(s)$, for the circuit in Figure 2.6(*b*). Use nodal analysis.

SOLUTION:

For this problem, we sum currents at the nodes rather than sum voltages around the meshes. From Figure 2.6(*b*) the sum of currents flowing from the nodes marked $V_L(s)$ and $V_C(s)$ are, respectively,

$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0 \qquad \textbf{2.35a}$$

$$CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0 \qquad \textbf{2.35b}$$

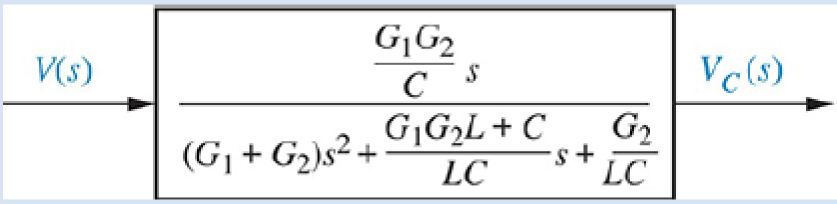
Rearranging and expressing the resistances as conductances,5 $G_1 = 1/R_1$ and $G_2 = 1/R_2$, we obtain

$$\left(G_1 + G_2 + \frac{1}{Ls}\right)V_L(s) \qquad - G_2V_C(s) = V(s)\,G_1 \qquad \textbf{2.36a}$$

$-G_2V_L(s) + (G_2 + Cs)V_C(s) = 0$ **2.36b**

Solving for the transfer function, $V_C(s)/V(s)$, yields Eq. (2.37) as shown in Figure below.

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1G_2}{C}s}{(G_1+G_2)s^2 + \frac{G_1G_2L+C}{LC}s + \frac{G_2}{LC}} \qquad \textbf{2.37}$$



Block diagram of the network