Tikrit university

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Control Engineering

Chapter 2

Transfer functions

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2.1 Laplace Transform Review

A system represented by a differential equation is difficult to model as a block diagram. Thus, we now lay the groundwork for the Laplace transform, with which we can represent the input, output, and system as separate entities. Further, their interrelationship will be simply algebraic. Let us first define the Laplace transform and then show how it simplifies the representation of physical systems (*Nilsson,1996*). The Laplace transform is defined as

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)\, e^{-st} dt \qquad \textbf{2.1}$$

where $s = \sigma + j\omega$, a complex variable. Thus, knowing $f(t)$ and that the integral in Eq. (2.1) exists, we can find a function, $F(s)$, that is called the *Laplace transform* of $f(t)$. The notation for the lower limit means that even if $f(t)$ is discontinuous at $t = 0$, we can start the integration prior to the discontinuity as long as the integral converges. Thus, we can find the Laplace transform of impulse functions. This property has distinct advantages when applying the Laplace transform to the solution of differential equations where the initial conditions are discontinuous at $t = 0$. Using differential equations, we have to solve for the initial conditions after the discontinuity knowing the initial conditions before the discontinuity. Using the Laplace transform we need only know the initial conditions before the discontinuity. The inverse Laplace transform, which allows us to find $f(t)$ given $F(s)$, is

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st} ds = f(t)\, u(t) \qquad \textbf{2.2}$$

where

$u(t) = 1 \quad t > 0$

$\quad\;\; = 0 \quad t < 0$

is the unit step function. Multiplication of f(t) by u(t) yields a time function that is zero for t < 0.

Using Eq. (2.1), it is possible to derive a table relating f(t) to F(s) for specific cases. Table 2.1 shows the results for a representative sample of functions. If we use the tables, we do not have to use Eq. (2.2), which requires complex integration, to find f(t) given F(s). In the following example we demonstrate the use of Eq. (2.1) to find the Laplace transform of a time function.

TABLE 2.1

Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Example 2.1 Laplace Transform of a Time Function

PROBLEM:

Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

SOLUTION:

Since the time function does not contain an impulse function, we can replace the lower limit of Eq. (2.1) with 0. Hence,

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^\infty Ae^{-at}e^{-st}\,dt = A\int_0^\infty e^{-(s+a)t}\,dt = -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a} \tag{2.3}$$

TABLE 2.2

Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}\left[e^{-at}f(t)\right] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

1. For this theorem to yield correct finite results, all roots of the denominator of F(s) must have negative real parts, and no more than one can be at the origin.

2. For this theorem to be valid, f(t) must be continuous or have a step discontinuity at t = 0 (i.e., no impulses or their derivatives at t = 0). In addition to the Laplace transform table, Table 2.1, we can use Laplace transform theorems, listed in Table 2.2, to assist in transforming between f(t) and F(s). In the next example, we demonstrate the use of the Laplace transform theorems shown in Table 2.2 to find f(t) given F(s).

Example 2.2 Inverse Laplace Transform

PROBLEM:

Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

SOLUTION:

For this example we make use of the frequency shift theorem, Item 4 of Table 2.2, and the Laplace transform of $f(t) = tu(t)$, Item 3 of Table 2.1. If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$, the inverse transform of $F(s+a) = 1/(s+a)^2$ is $e^{-at}tu(t)$. Hence, $f_1(t) = e^{-3t}tu(t)$.

Partial-Fraction Expansion

To find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term. The result is called a partial-fraction expansion. If F1(s) = N(s)/D(s), where the order of N(s) is less than the order of D(s), then a partial-fraction expansion can be made. If the order of N(s) is greater than or equal to the order of D(s), then N(s) must be divided by D(s) successively until the result has a remainder whose numerator is of order less than its denominator. For example, if

$$F_1(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5} \tag{2.4}$$

we must perform the indicated division until we obtain a remainder whose numerator is of order less than its denominator. Hence,

$$F_1(s) = s + 1 + \frac{2}{s^2 + s + 5} \tag{2.5}$$

Taking the inverse Laplace transform, using Item 1 of Table 2.1, along with the differentiation theorem (Item 7) and the linearity theorem (Item 3 of Table 2.2), we obtain

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathscr{L}^{-1}\left[\frac{2}{s^2+s+5}\right] \quad \textbf{2.6}$$

Using partial-fraction expansion, we will be able to expand functions like $F(s) = 2/(s^2 + s + 5)$ into a sum of terms and then find the inverse Laplace transform for each term. We will now consider three cases and show for each case how an $F(s)$ can be expanded into partial fractions.

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

An example of an F(s) with real and distinct roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)} \quad \textbf{2.7}$$

The roots of the denominator are distinct, since each factor is raised only to unity power. We can write the partial-fraction expansion as a sum of terms where each factor of the original denominator forms the denominator of each term, and constants, called ***residues***, form the numerators. Hence,

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)} \quad \textbf{2.8}$$

To find K1, we first multiply Eq. (2.8) by (s + 1), which isolates K1. Thus,

$$\frac{2}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)} \quad \textbf{2.9}$$

Letting s approach −1 eliminates the last term and yields $K1 = 2$. Similarly, $K2$ can be found by multiplying Eq. (2.8) by $(s + 2)$ and then letting s approach −2; hence, $K2 = -2$. Each component part of Eq. (2.8) is an $F(s)$ in Table 2.1. Hence, $f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = \left(2e^{-t} - 2e^{-2t}\right)u(t) \quad \textbf{2.10}$$

Example 2.3 Laplace Transform Solution of a Differential Equation

PROBLEM:

Given the following differential equation, solve for *y*(*t*) if all initial conditions are zero. Use the Laplace transform

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t) \qquad 2.11$$

SOLUTION:

Substitute the corresponding *F*(*s*) for each term in Eq. (2.11), using Item 2 in Table 2.1, Items 7 and 8 in Table 2.2, and the initial conditions of *y*(*t*) and *dy*(*t*)/*dt* given by *y* (0 −) = 0 and y (0−) = 0, respectively. Hence, the Laplace transform of Eq. (2.11) is

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

Solving for the response, Y(s), yields

$$Y(s) = \frac{32}{s(s^2+12s+32)} = \frac{32}{s(s+4)(s+8)}$$

To solve for *y*(*t*), we notice that Eq. does not match any of the terms in Table 2.1. Thus, we form the partial-fraction expansion of the right-hand term and match each of the resulting terms with *F*(*s*) in Table 2.1. Therefore,

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1 \qquad K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2 \qquad K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1$$

Hence, $Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)}$ Since each of the three component parts of Eq. is represented as an *F*(*s*) in Table 2.1, *y*(*t*) is the sum of the inverse Laplace transforms of each term. Hence, $y(t) = \left(1 - 2e^{-4t} + e^{-8t}\right)u(t)$

Case 2. Roots of the Denominator of *F*(*s*) Are Real and Repeated

Example 2.4 F(s) with real and repeated roots in the denominator is

$F(s)=\frac{2}{(s+1)(s+2)^2}$

SOLUTION:

The roots of $(s + 2)^2$ in the denominator are repeated, since the factor is raised to an integer power higher than 1. In this case, the denominator root at −2 is a *multiple root* of *multiplicity* 2. We can write the partial-fraction expansion as a sum of terms, where each factor of the denominator forms the denominator of each term. In addition, each multiple roots generates additional terms consisting of denominator factors of reduced multiplicity.

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

then $A = 2$, which can be found as previously described. B can be isolated by multiplying Eq. (2.23) by $(s + 2)^2$, yielding

$$\frac{2}{s+1} = \frac{(s+2)^2 A}{s+1} + B + C(s+2)$$

Letting s approach −2, $B = -2$. To find C we see that if we differentiate above equation with respect to s,

$$\frac{-2}{(s+1)^2} = \frac{(s+2)sA}{(s+1)^2} + C$$

C is isolated and can be found if we let s approach −2. Hence, $C = -2$.

$$F(s) = \frac{2}{s+1} + \frac{-2}{(s+2)^2} + \frac{-2}{s+2}$$

$F(s)$ in Table 2.1; hence, $f(t)$ is the sum of the inverse Laplace

$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

Example 2.5 F(s) *F*(*s*) with complex roots in the denominator is

$F(s)=\frac{3}{s\,(s^2+2s+5)}$

SOLUTION:

This function can be expanded in the following form:

$$F(s)=\frac{3}{s\,(s^2\ +\ 2s\ +\ 5)}\quad=\quad\frac{A}{s}+\frac{Bs+C}{(s^2\ +\ 2s\ +\ 5)}$$

A is found in the usual way to be. B and C can be found by first multiplying Eq. by t lowest common denominator, s(s2 + 2s + 5), and clearing the fractions. After simplificati with A = $\frac{3}{5}$, we obtain $3\ =\ \left(B+\frac{3}{5}\right)S^2+\left(C+\frac{6}{5}\right)S+3$

Balancing coefficients $\left(B+\frac{3}{5}\right)=0$ and $\left(C+\frac{6}{5}\right)=0$ Hence B=$\frac{-3}{5}$, C=$\frac{-6}{5}$ Thus

The last term can be shown to be the sum of the Laplace transforms of an exponentially damped sine and cosine. Using Item 7 in Table 2.1 and Items 2 and 4 in Table 2.2, we get

$\mathscr{L}$ [Ae^{-at} cos ωt]= $\frac{A(s+a)}{(s+a)^2+W^2}$ *and* $\mathscr{L}$ [Be^{-at} sin ωt]= $\frac{BW}{(s+a)^2+W^2}$

Adding two equations above we get $\frac{A(s+a)+BW}{(s+a)^2+W^2}$ $\Rightarrow F(s)=\frac{3}{s\,(s^2+2s+5)}=\frac{\frac{3}{5}}{s}-\frac{\frac{3}{5}(s+2)}{s^2+2s+5}$

$$F(s)=\frac{\frac{3}{5}}{s}-\frac{3}{5}\,\frac{(s+1)+\frac{1}{2}2}{(s+1)^2+2^2}$$

f (t) $=\frac{3}{5}-\frac{3}{5}e^{-t}\left(\cos 2t+\frac{1}{2}\sin 2t\right)$