Properties are important features of transform. They make the transform easier to work with, and thus extend the transform's utility. Since transform all do the same thing (i.e. decompose a general signal into a linear combination of basic signals), properties for different transforms are often similar. So learning a set of properties for one transform directly contributes to understanding and employing other transforms.

Symmetry: For real-valued periodic *x(t),* with FS coefficients a_k, we have that

$$a_{-k} = a_k^* \ .$$

That is, the CT FS coefficients are symmetric

Linearity: Consider two signals, $x_1(t)$ and $x_2(t)$, each periodic with period T , with CTFS coefficients a_{k1}, a_{k2} respectively. Given any two constants c_1 and c_2, the CTFS pair is

$$c_1\, x_1(t) + c_2 x_2(t) \quad \longleftrightarrow \quad c_1\, a_{k1} + c_2\, a_{k2}$$

Time Shift *(i.e. Delay):* Given a signal *x(t),* periodic with period T , with CTFS coefficients a_k, then for any delay τ, $x(t-\tau)$ is also periodic with period T with CTFS pair

$$x(t-\tau) \quad \longleftrightarrow \quad e^{-jk\omega_0\tau}\, a_k$$

Note that here the magnitude of the series coefficients are not changed, only their phases.

Discrete Time Fourier Series (DTFS)

The discrete Fourier series representation of a periodic sequence *x[n]* with fundamental period N_o (N_o is the number of data points in the data set) is given by

$$x[n] = \sum_{k=0}^{N_0-1} C_k e^{jk\Omega n}$$

$$\Omega = \frac{2\pi}{N_0}$$

where C_k, are the Fourier coefficients and given by

$$C_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-jk\Omega n}$$

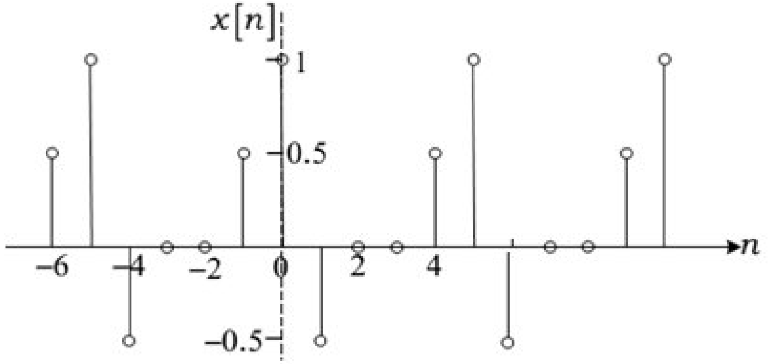
The DTFS coefficients C_k are called frequency-domain representation for *x[n]* since each coefficient is associated with a complex sinusoid of a different frequency. Setting $k = 0$,

$$C_0 = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]$$

which indicates that c_0 equals the average value of ***x[n]*** over a period. The Fourier coefficients C_k are often referred to as the **spectral coefficients** of ***x[n]***. The Fourier series coefficients C_k are periodic with fundamental period N_0.

$$C_{k+N_0} = C_k$$

Example: Determine the DTFS for the following signal



Solution

The signal has a period N = 5, hence $\Omega_0 = 2\pi/5$. Also the signal has odd symmetry, hence we can sum over n=-2 to n =2 in Eq. below to get

$$C_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-jk\Omega n} \quad c_k = \frac{1}{5}\sum_{n=-2}^{2} x[n]\; e^{-jk\frac{2\pi n}{5}}$$

$$= \frac{1}{5}\left\{x[-2]e^{j4\pi k/5} + x[-1]e^{j2\pi k/5} + x[0]e^{j0} + x[1]e^{-j2\pi k/5} + x[2]e^{-j4\pi k/5}\right\}$$

Substituting for x***[n]*** from above Fig., we get

$$c_k = \frac{1}{5}\left\{0 + \frac{1}{2}e^{j2\pi k/5} + 1e^{j0} - \frac{1}{2}e^{-j2\pi k/5} + 0\right\}$$

$$= \frac{1}{5}\left\{1 + \frac{1}{2}e^{j2\pi k/5} - \frac{1}{2}e^{-j2\pi k/5}\right\}$$

$$= \frac{1}{5}\left\{1 + j\sin(2\pi k/5)\right\}$$

From this equation, one period of the DTFS coefficients c_k for k =-2 to k = 2 are

$$c_{-2} = c[-2] = \frac{1}{5}\{1 - j\sin(4\pi/5)\} = 0.2 - j0.1176 = 0.232\angle -0.5315\,rad$$

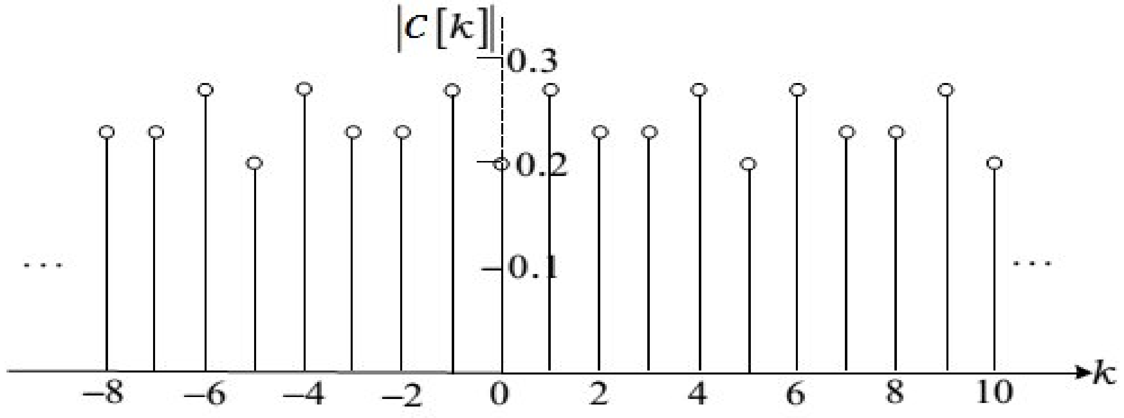
$$c_{-1} = c[-1] = \frac{1}{5}\{1 - j\sin(2\pi/5)\} = 0.2 - j0.1902 = 0.276\angle -0.76\,rad$$

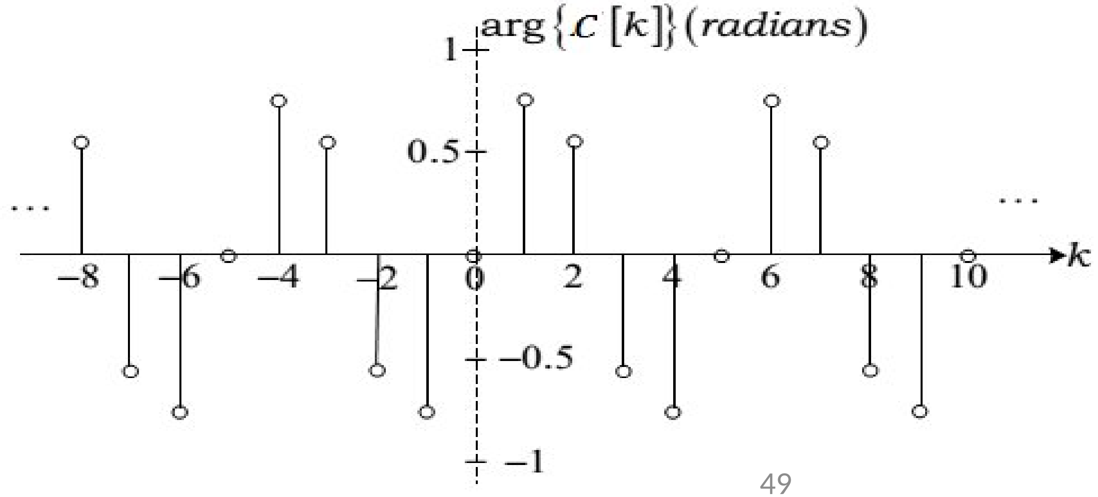
$$c_0 = c[0] = \frac{1}{5}\{1 - j\sin(0)\} = 0.2 = 0.2\angle 0\,rad$$

$$c_1 = c[1] = \frac{1}{5}\{1 + j\sin(2\pi/5)\} = 0.2 + j0.1902 = 0.276\angle 0.76\,rad$$

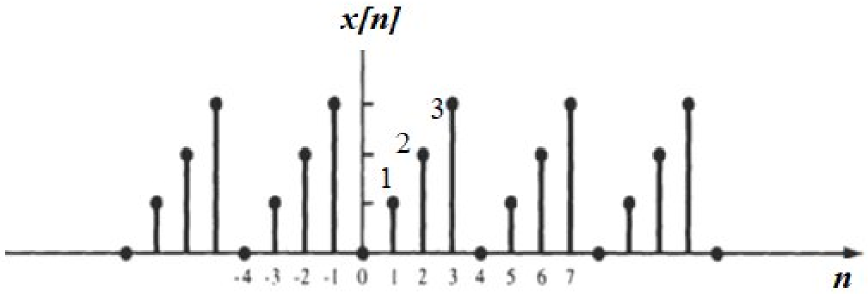
$$c_2 = c[2] = \frac{1}{5}\{1 + j\sin(4\pi/5)\} = 0.2 + j0.1176 = 0.232\angle 0.5315\,rad$$

Hence, the magnitude and phase spectrums of C_k are as shown





Example Determine the Fourier coefficients for the periodic sequence ***x[n]*** shown in Figure



Solution:

$$C_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-jk\Omega n}$$

From Fig. we see that $x[n]$ is the periodic extension of $\{0, 1, 2, 3\}$ with fundamental period $N_0 = 4$. Thus,

$$\Omega_0 = \frac{2\pi}{4} \qquad \text{and} \qquad e^{-j\Omega_0} = e^{-j2\pi/4} = e^{-j\pi/2} = -j$$

the discrete-time Fourier coefficients c_k are

$$C_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\,(-j)^{kn}$$

$$c_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = \frac{1}{4}(0+1+2+3) = \frac{3}{2}$$

$$c_1 = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^n = \frac{1}{4}(0-j1-2+j3) = -\frac{1}{2}+j\frac{1}{2}$$

$$c_2 = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^{2n} = \frac{1}{4}(0-1+2-3) = -\frac{1}{2}$$

$$c_3 = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^{3n} = \frac{1}{4}(0+j1-2-j3) = -\frac{1}{2}-j\frac{1}{2}$$

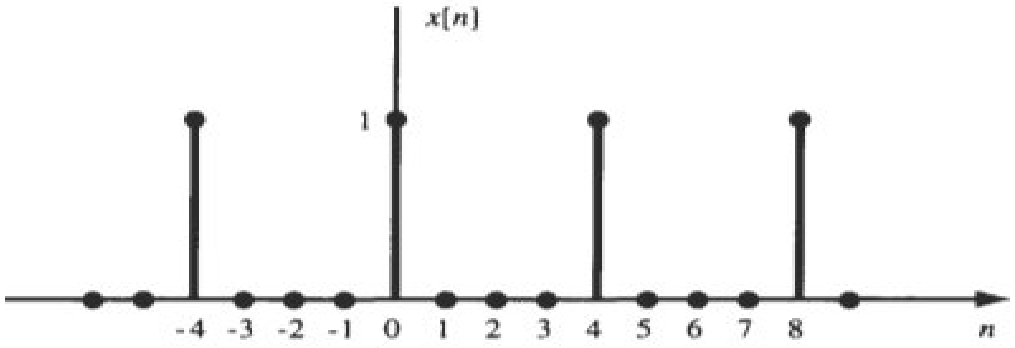
Note that $c_3 = c_{4-1} = c_1^*$

Example Consider a sequence

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$$

(*a*) Sketch $x[n]$.

(*b*) Find the Fourier coefficients c_k of $x[n]$.



(*a*)

Solution

(*a*) The sequence $x[n]$ is sketched in Fig. It is seen that $x[n]$ is the periodic extension of the sequence $\{1, 0, 0, 0\}$ with period $N_0 = 4$.

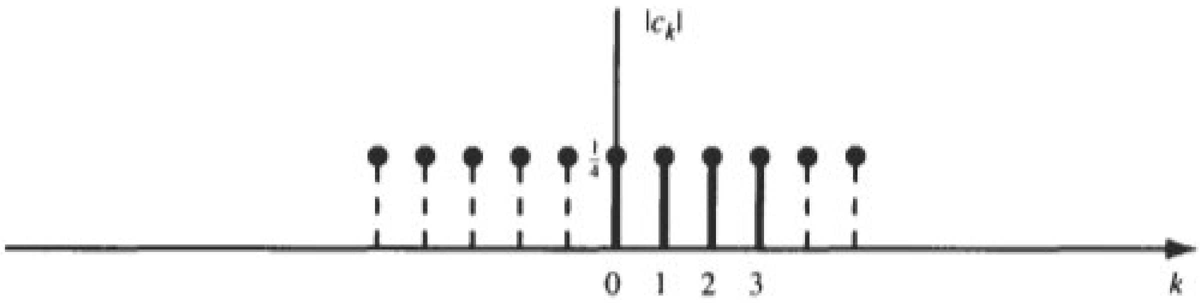
(*b*)

$$x[n] = \sum_{k=0}^{3} c_k e^{jk(2\pi/4)n} = \sum_{k=0}^{3} c_k e^{jk(\pi/2)n}$$

and

$$c_k = \frac{1}{4}\sum_{n=0}^{3} x[n] e^{-jk(2\pi/4)n} = \frac{1}{4}x[0] = \frac{1}{4} \qquad \text{all } k$$

since $x[1] = x[2] = x[3] = 0$. The Fourier coefficients of $x[n]$ are sketched



Properties of discrete Fourier series

***Periodicity of Fourier Coefficients*:** The Fourier series coefficients c_k are periodic with fundamental period N_0.

$$C_{k+N_0} = C_k$$

Symmetry or Duality: Symmetry property of the discrete Fourier series is given by

$$x[n] \xleftrightarrow{\text{DFS}} c_k = c[k] \qquad c[n] \xleftrightarrow{\text{DFS}} \frac{1}{N_0} x[-k]$$

Even and Odd Sequences:

When $x[n]$ is real, let

$$x[n] = x_e[n] + x_o[n]$$

where $x_e[n]$ and $x_o[n]$ are the even and odd components of $x[n]$, respectively. Let

$$x[n] \xleftrightarrow{\text{DFS}} c_k$$

Then

$$x_e[n] \xleftrightarrow{\text{DFS}} \text{Re}[c_k]$$

$$x_o[n] \xleftrightarrow{\text{DFS}} j\,\text{Im}[c_k]$$

Thus, we see that if $x[n]$ is real and even, then its Fourier coefficients are real, while if $x[n]$ is real and odd, its Fourier coefficients are imaginary.

***Other Properties**:* When $x[n]$ is real, then it follows that

$$c_{-k} = c_{N_0-k} = c_k^*$$

If ***x[n]*** is represented by the discrete Fourier series, then it can be shown that

$$\frac{1}{N_0}\sum_{n=\langle N_0\rangle}|x[n]|^2=\sum_{k=\langle N_0\rangle}|c_k|^2$$

which is called Parseval's identity (or Parseval's theorem) for the discrete Fourier series.

Example The discrete-time Fourier representation of a periodic signal $x[n]=\{1,1,0,0\}$ with period $N=4$ is given by,

$$c_k=\frac{1}{4}\sum_{n=0}^{3}x[n]e^{-j2\pi kn/4}=\frac{1}{4}(1+e^{-j2\pi k/4}) \qquad k=0,1,2,3$$

This gives the coefficients

$$c_0=\frac{1}{2}; \qquad c_1=\frac{1}{4}(1-j); \qquad c_2=0; \qquad c_3=\frac{1}{4}(1+j)$$

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1}|x[n]|^2=\sum_{n=0}^{N_0-1}|C_k|^2=\frac{1}{2}$$

Note that $|a\pm jb|=\sqrt{a^2+b^2}$

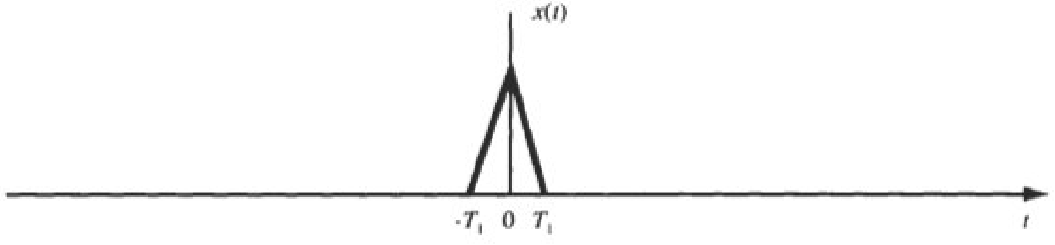
The Fourier transform is used for mapping between the frequency and time domains. The Fourier transform is a major tool in system analysis and consequently in simulation of systems.

From Fourier Series to Fourier Transform:

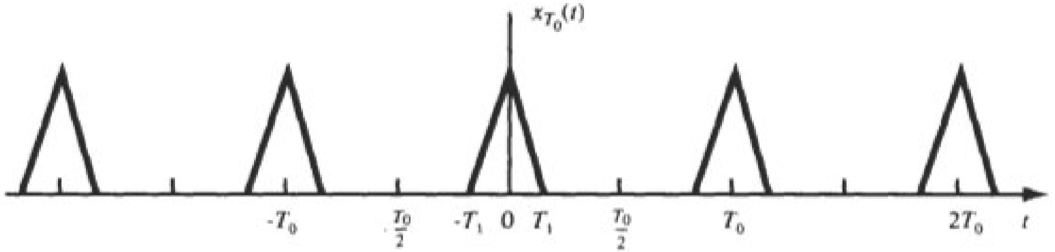
Fourier series is for periodic signals. Fourier transform is for non-periodic signals. Let ***x(t)*** be a nonperiodic signal of finite duration, that is,

$$x(t) = 0 \qquad |t| > T_1$$

Such a signal is shown in Figure a .Let $x_{T0}(t)$ be a periodic signal formed by repeating $x(t)$ with fundamental period T_0 as shown in Figure b.

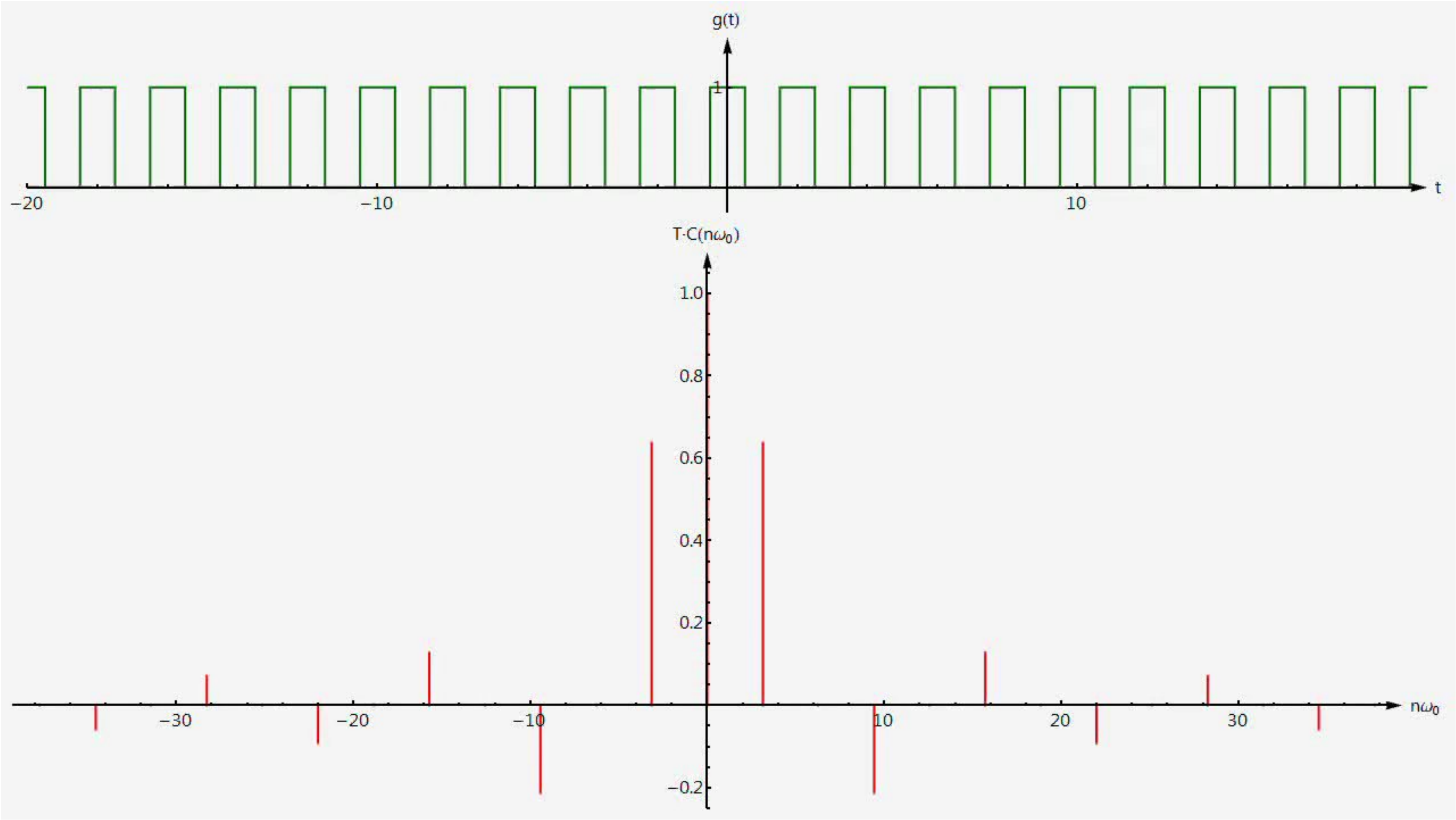


(a)



(b)

From Fourier Series to Fourier Transform:



When the period of $x_T(t)$ approaches infinity, the periodic signal $x_T(t)$ becomes a non-periodic signal $x(t)$ and the following will result:

1- Interval between two neighbouring frequency components becomes zero: $T \to \infty \Longrightarrow \omega_0 = 2\pi/T \to 0$

2- Discrete frequency becomes continuous frequency $k\omega_0|_{\omega_0 \to 0} \Longrightarrow \omega$

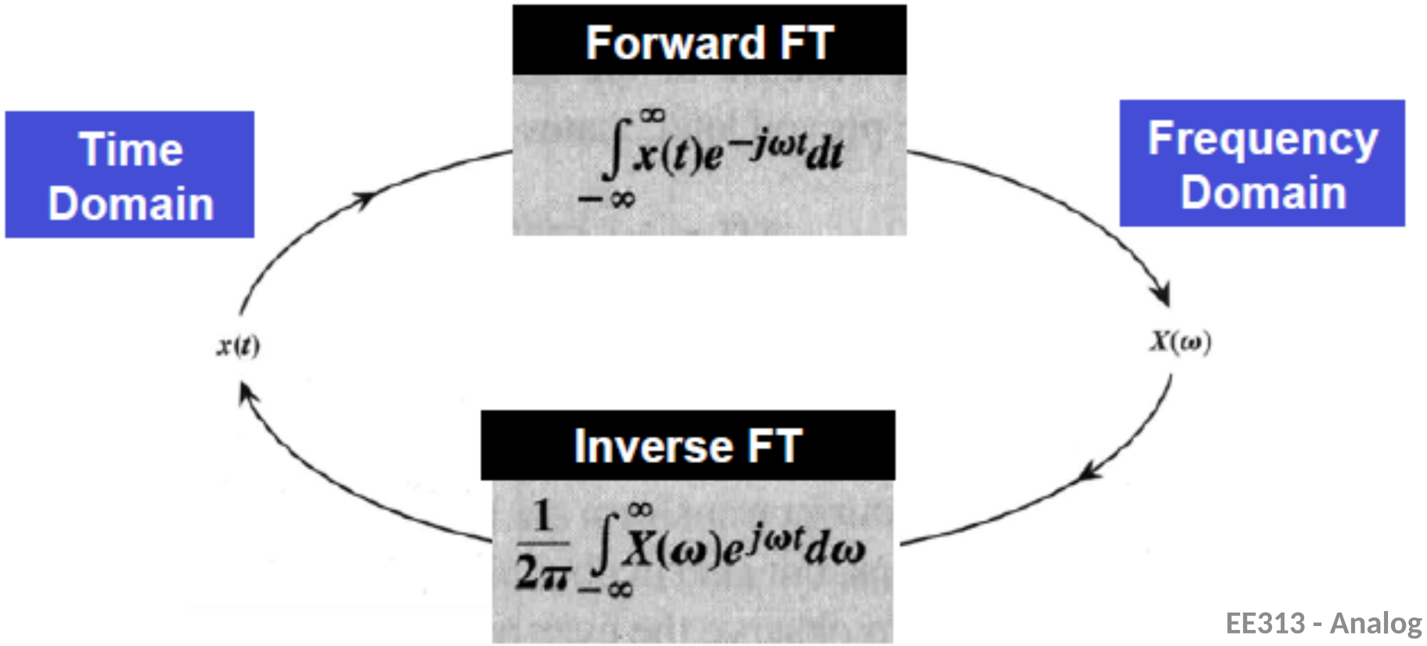
3- Summation of the Fourier expansion becomes an integral:

In summary, the Fourier transform of *x(t)* can be written as

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

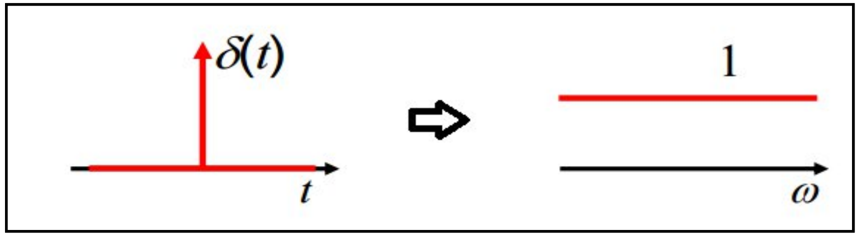
The inverse Fourier transform of ***X(ω)*** is denoted by

$$x(t) = \mathscr{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$



Example: find FT for $x(t)=\delta(t)$

Solution: $X(\omega)=\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = e^{-j\omega(0)} = 1$

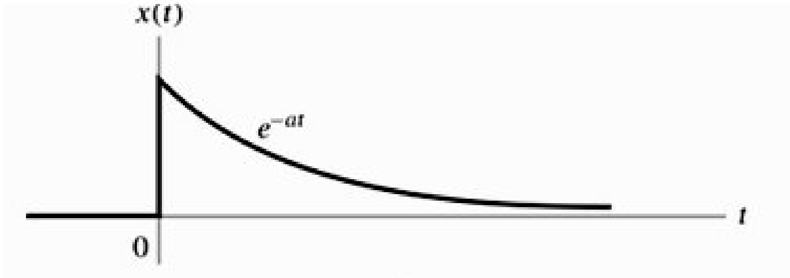


Example: Find the FT of $x(t) = e^{-at}\, u(t)$. plot the magnitude and phase spectrum.

Solution: let us find the FT of $x(t) = e^{-at}\, u(t)$ for $a > 0$,

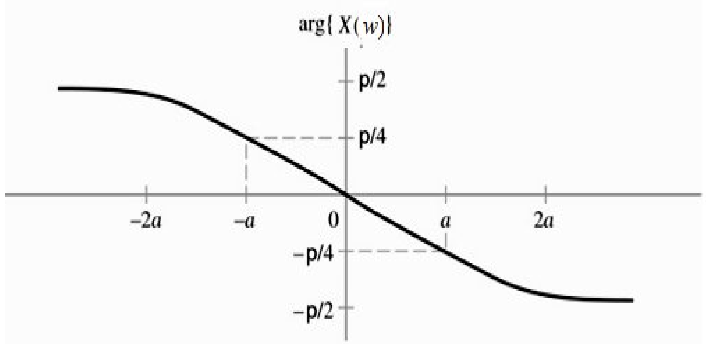
$$X(\omega) = \int_0^{\infty} e^{-at} u(t) e^{-j\omega t}\, dt$$

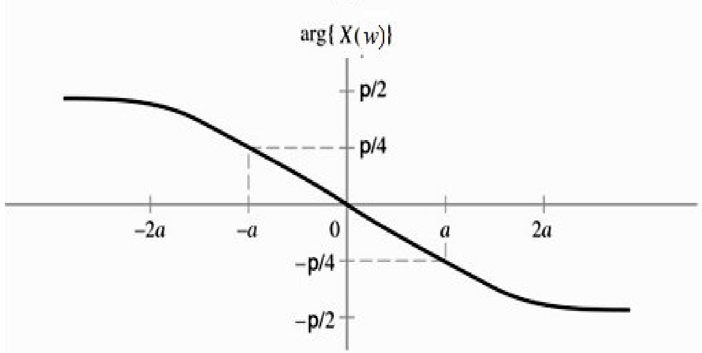
$$= \int_0^{\infty} e^{-(a+j\omega)t}\, dt = -\frac{1}{a+j\omega} e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$



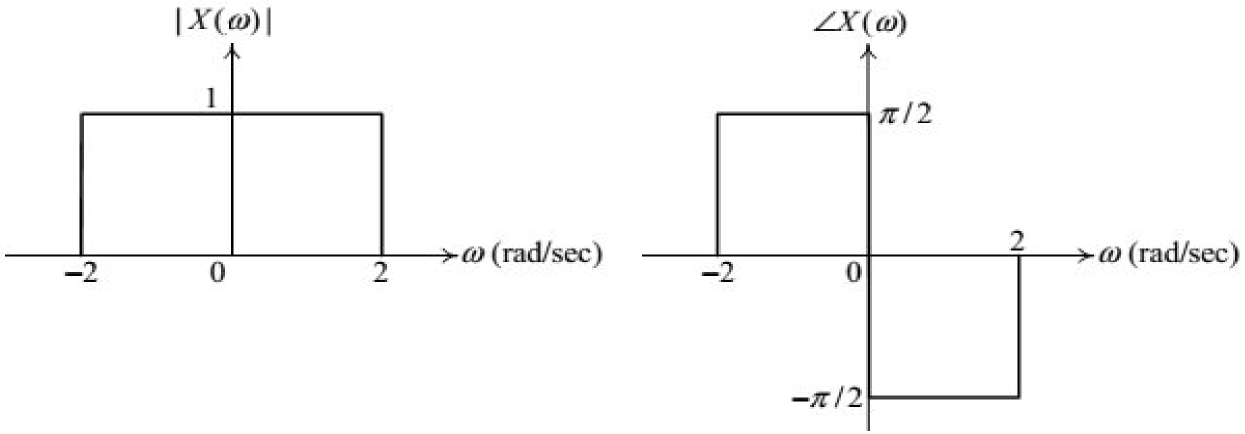
Converting to polar form, we find that the magnitude and phase of $X(\omega)$ are respectively given by

$$|X(\omega)| = \frac{1}{(a^2+\omega^2)^{\frac{1}{2}}} \quad and \quad arg\{X(\omega)\} = -\tan^{-1}\left(\frac{\omega}{a}\right)$$





Example: Determine the continuous-time signal *x(t)* if its magnitude and phase spectra are shown below



Solution

The Fourier transform ***X(ω)*** is expressed mathematically as:

$$X(\omega)=\begin{cases} e^{j\frac{\pi}{2}}, & -2 \le \omega \le 0 \\ e^{-j\frac{\pi}{2}}, & 0 \le \omega \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Using the inverse Fourier transform, the continuous-time function can be found as:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\left[\int_{-2}^{0} e^{j\frac{\pi}{2}} e^{j\omega t} d\omega + \int_{0}^{2} e^{-j\frac{\pi}{2}} e^{j\omega t} d\omega\right] \\ &= \frac{1}{2\pi}\left[\frac{1}{t} e^{j\omega t}\Big|_{-2}^{0} - \frac{1}{t} e^{j\omega t}\Big|_{0}^{2}\right] = \frac{1}{2\pi t}\left[1 - e^{-j2t} - e^{j2t} + 1\right] \\ &= \frac{1}{2\pi t}\left[2 - 2\cos(2t)\right] = \frac{1}{\pi t}\left[1 - \cos(2t)\right] \end{aligned}$$

Since:

- $e^{j\frac{\pi}{2}} = j$
- $e^{-j\frac{\pi}{2}} = -j$

Basic properties of the Fourier transform are presented in the following.

Linearity: $a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$

Time Shifting: $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$

Frequency Shifting: $e^{j\omega_0 t} x(t) \leftrightarrow X(\omega - \omega_0)$

The multiplication of $x(t)$ by a complex exponential signal $e^{j\omega_0 t}$ is sometimes called *complex modulation*. Thus, Eq. above shows that complex modulation in the time domain corresponds to a shift of $X(\omega)$ in the frequency domain. Note that the frequency-shifting property Eq. above is the dual of the time-shifting property

Time Scaling: $x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

where a is a real constant. This property follows directly from the definition of the Fourier transform. Equation above indicates that scaling the time variable t by the factor a causes an inverse scaling of the frequency variable ω by $1/a$, as well as an amplitude scaling of $X(\omega/a)$ by $1/|a|$. Thus, the scaling property above implies that time compression of a signal ($a > 1$) results in its spectral expansion and that time expansion of the signal ($a < 1$) results in its spectral compression.

Time Reversal: $x(-t) \leftrightarrow X(-\omega)$

Thus, time reversal of $x(t)$ produces a like reversal of the frequency axis for $X(\omega)$. This property is readily obtained by setting $a = -1$ in time scaling Eq.

Duality (or Symmetry): $X(t) \leftrightarrow 2\pi x(-\omega)$

The duality property of the Fourier transform has significant implications. This property allows us to obtain both of these dual Fourier transform pairs from one evaluation of Eq. below

Differentiation in the Time Domain: $\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

This shows that the effect of differentiation in the time domain is the multiplication of X(ω) by ***jω*** in the frequency domain.

Differentiation in the Frequency Domain: which is the dual property of differentiation in the time domain:

$$(-jt)x(t) \leftrightarrow \frac{dX(\omega)}{d\omega} \quad \text{or } \mathscr{F}\{t\, x(t)\} = j\frac{d}{d\omega}X(\omega)$$

Integration in the Time Domain: $\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \pi X(0)\,\delta(\omega) + \frac{1}{j\omega}X(\omega)$

Since integration the inverse of differentiation, this Eq. shows that the frequency-domain operation corresponding to time-domain integration is multiplication by $1/j\omega$, but an additional term is needed to account for a possible dc component in the integrator output. Hence, unless $X(0) = 0$, a dc component is produced by the integrator

Convolution: $x_1(t) * x_2(t) \leftrightarrow X_1(\omega) X_2(\omega)$

Equation above is referred to as the time convolution theorem, and it states that convolution in the time domain becomes multiplication in the frequency domain.